Special Functions
Calculus II Project

The purpose of this project is to explore some familiar functions as well as a couple of functions that, although well documented, are rarely encountered in intermediate mathematics study. In the study of calculus, we do encounter somewhat exotic functions whose usefulness seems obvious (e.g. fractional powers, trigonometric, exponential, etc.), and whose properties become familiar to the point of losing any sense of mystery.

Herein you will consider alternative approaches to defining and deriving some common functions. Some of the familiar properties will also be obtained using the new perspective. Some (likely new to you) functions will also be defined and analyzed.

Carry out the following activities.

A. The exponential function e^x is first encountered as an extension of the simple process of *repeated addition*. Of course this view breaks down when considering the expression $e^{1/2}$, but this is easily rectified by extension of the property $(a^b)^c = a^{bc}$. The expression e^π is a little tougher to reconcile since π is irrational. Never the less, we accept e^π as meaningful. A *calculus* based approach to deriving the exponential due to Jacob Bernoulli involves taking the limit as the frequency of compounding interest tends to infinity. An alternative approach to deriving e^x and some of its properties is obtained by seeking a (nonzero) solution to the differential equation

$$\frac{dy}{dx} = y.$$

Suppose a nonzero solution exists in the form $y = \sum_{n=0}^{\infty} c_n x^n$. Also assume that the power rule can be used in the ordinary way to differentiate the sum term by term. Use this to derive the relation

$$c_{n+1} = \frac{c_n}{n+1}, \quad n \geq 0$$

and hence $y = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

B. Suppose the series $\sum_{n=0}^{\infty} a_n$ converges to A and $\sum_{k=0}^{\infty} b_k$ converges to B (with at least one of the convergences being absolute). Then

$$AB = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} b_k\right).$$

This is known as the Cauchy product. (You can take it on faith or readily find a proof in the literature.)

Use this to show that the summation representation for e^x obtained in part **A.** satisfies the well known property

$$e^{x+z} = e^x e^z.$$

C. Derive the sine and cosine functions without reference to angles or geometry. Recall that the imaginary unit i is the principal square root of -1; $i = \sqrt{-1}$. Consider the pair of functions defined on $(-\infty, \infty)$

$$S(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \text{and} \quad C(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

- Show that $S(x)$ is an odd function, and $C(x)$ is an even function.
- Show that $[C(x)]^2 + [S(x)]^2 = 1$ for all real x.
- Show that $S'(x) = C(x)$, $C'(x) = -S(x)$, and hence both S and C solve the differential equation
$$\frac{d^2y}{dx^2} + y = 0.$$
- Using the summation formula for e^x from part **A.** derive summation formulas for $S(x)$ and $C(x)$. (Do the algebra necessary to eliminate all traces of the imaginary unit from the results.)
- Identify S and C by their more common names. Use this to obtain an expression for e^{ix} and to evaluate $e^{i\pi}$.

D. Define a pair of functions similar to those in part **C.**

$$S_h(x) = \frac{e^x - e^{-x}}{2}, \quad \text{and} \quad C_h(x) = \frac{e^x + e^{-x}}{2}.$$

- Show that $S_h(x)$ is an odd function, and $C_h(x)$ is an even function.
- Show that $[C_h(x)]^2 - [S_h(x)]^2 = 1$ for all real x.
- Show that $S_h'(x) = C_h(x)$, $C_h'(x) = S_h(x)$, and hence both S_h and C_h solve the differential equation
$$\frac{d^2y}{dx^2} - y = 0.$$
- Analyze these functions in terms of their increasing/decreasing behavior, concavity, symmetry, intercepts, etc. and obtain plots.
- Find these functions in the literature to identify them by their common names.

E. In *Late Transcendentals*, the natural logarithm is defined via an integral. This also defines an antiderivative for the function x^{-1} (unobtainable merely by reference to power functions). Consider the similar difficulty that arises from the seemingly modest expression $\int e^{-x^2}\,dx$.

Solve the differential equation

$$\frac{d}{dx}\left(e^{x^2}\frac{dy}{dx}\right) = 0 \quad \text{subject to} \quad y'(0) = \frac{2}{\sqrt{\pi}}, \quad y(0) = 0.$$

(You can do this by integrating. Your final result will have to be stated as an integral in the tradition of definition [1] on page 421 in Stewart.)

The solution is usually denoted as erf(x) and is referred to as the *error function*.

- Express the indefinite integral $\int e^{-x^2}\,dx$ in terms of erf(x).
- Show that erf(x) is an odd function.
- Show that erf(x) is increasing on $(-\infty, \infty)$ and has one point of inflection at the origin.
- It is known that $\int\limits_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$. Use this to show that

$$\lim_{x\to\infty} \text{erf}(x) = 1.$$

F. The Gamma function, denoted $\Gamma(x)$, is defined by[1]

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0.$$

- Evaluate $\Gamma(x)$ for $x = 1, 2,$ and 3.
- Show that for $x > 0$, $\Gamma(x+1) = x\Gamma(x)$. (You can do this with your current knowledge of integration techniques.)
- Using your value of $\Gamma(1)$ and the relation derived to show that for integers $n \geq 1$, $n! = \Gamma(n+1)$. Is it clear that $0! = 1$ is more than a convenient convention?
- Use the fact that for $x \approx 0$, $e^{-t} \approx 1$ to show that $\Gamma(0)$ isn't defined (the integral is divergent). In particular, $\Gamma(x) \to \infty$ as $x \to 0^+$.

[1]The domain can be extended to all reals excluding nonpositive integers or even all complex numbers excluding nonpositive integers. Here we consider the domain of interest to be $(0, \infty)$.