Taylor Polynomials

Calculus I Project

The purpose of this project is to find a systematic way to approximate exotic function by polynomials. Polynomials have the advantage of being easy to evaluate (requiring only multiplication and addition/subtraction). We know that if a function f is differentiable at a point a, then the tangent line L at a approximates f at points very near to a. Unfortunately, a line has no *bendiness* to it. so the tangent line can only capture the slope of a function, not concavity or other features.

Let's observe that if f has tangent line L at a, then

$$L(a) = f(a) \quad \text{and} \quad L'(a) = f'(a).$$

So L and its derivative of a match those of f. A natural extension of this is to find a polynomial P that satisfies three criteria

$$P(a) = f(a), \quad P'(a) = f'(a), \quad \text{and} \quad P''(a) = f''(a).$$

To get three conditions, we'll need three coefficients. So the lowest degree will have to be 2. Call P_2 the quadratic approximation to f at a.

A. Find a quadratic $P_2(x) = Ax^2 + Bx + C$ to approximate $f(x) = \sin(x) + \cos(x)$ near $a = 0$. Produce a plot of your polynomial P together with the tangent line L and f. Give a detailed commentary on what you see.

B. For a value of $a \neq 0$, it is advantageous to write P in the form

$$P_2(x) = A(x-a)^2 + B(x-a) + C. \tag{1}$$

Written as such, we typically say that P is *centered* at a. For any function f that is twice differentiable at a, find a formula for the coefficients A, B, and C so that P_2 is the quadratic approximation to f at a. (Expect your formulas to depend on f and its derivatives at the number a.)

C. Extend the idea to come up with four conditions that a cubic polynomial P_3 should satisfy to be considered a cubic approximation to a function f at a point a. Write out the general form for a cubic (in the manner of (1)), and for a sufficiently differentiable function f find formulas for the coefficients of your cubic.

D. Determine a cubic approximation to the function $f(x) = x\sqrt{x+1}$ at the point $a = 1$. Produce a plot of f together with P_3. Determine (using a calculator/computer or otherwise) an interval $\mathbf{I} = (1-\delta, 1+\delta)$ for which the error $|f(x) - P_3(x)| < 10^{-3}$ for every number x in $\mathbf{I}$.

E. Generalize the results to any degree polynomial n. That is, let P_n be a polynomial of degree n centered at a. Note that a general n^{th} degree polynomial centered at a will have the form

$$P_n(x) = b_n(x-a)^n + b_{n-1}(x-a)^{n-1} + \cdots + b_1(x-1) + b_0.$$

(You might want to experiment with some small numbers say P_4 and P_5 and then generalize letting n be any positive integer.) Suppose that a function f is sufficiently differentiable at a. Then determine formulas for the coefficients of your polynomial in terms of f and its various derivatives at a. (**Note:** You can streamline your notation by including factorials. If you're not familiar with factorials, do a search.)

F. Give a nice graphical example of your findings from part E. In particular, pick an interesting function (sines and cosines work well), and plot it along with several polynomial approximations (for example P_2, $P_3, \ldots, P_6$). The function f and center a can be of your choosing. Give a commentary on what you what you observe. Consider commenting on the advantages as well as any weaknesses to using your polynomial approximations.