Vector Valued Functions

Calculus II Project

In this project, you will investigate vector valued functions, curves, and some of their applications. In the plane, vector valued functions are an alternative view on parametric curves for which vector formalism can be used to express curves and their properties and uses. The extension to curves in a 3-dimensional space follows very naturally, and the vector operations in 3-dimensions provide compact formulations for characters of curves in space. Fortunately, the calculus of vector valued functions of a single variable only require familiarity with univariate calculus.

Carry out the following activities.

A. Review[1] and discuss the basics of vectors in $\mathbb{R}^2$. In particular, you should be familiar with (be able to use)

- a geometric and component form of vectors (including trigonometric and **i**, **j** notation),
- addition of vectors, scalar multiplication of vectors, and the properties of these operations,
- the norm (magnitude, length) of a vector,

You will also need to review the dot product. You should

- Define the dot product given vectors in component form.
- Connect the dot product to the angle between vectors and the vector norm.
- Define *orthogonality* with its geometric meaning.

B. A vector valued function in $\mathbb{R}^2$ is a function of a real variable whose output is a vector in $\mathbb{R}^2$. If we denote such a function $\mathbf{r}(t)$, then

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \quad \text{for real valued functions } f \text{ and } g.$$

The functions f and g are called the components of $\mathbf{r}$. Note that if (x, y) is the terminal point of $\mathbf{r}$, then this function corresponds to the parametric equations $x = f(t)$, $y = g(t)$. Unless the domain is specified, we take it to be the set of all t for which $\mathbf{r}$ is defined.

Produce plots of each of the vector valued functions

$$\text{(a)} \quad \mathbf{r}(t) = \cos t\mathbf{i} + 2t\mathbf{j}, \quad \text{(b)} \quad \mathbf{r}(t) = \frac{t-1}{\sqrt{t^2+1}}\mathbf{i} + \frac{t+1}{\sqrt{t^2+1}}\mathbf{j}$$

[1]The concepts in this activity are assumed to be prerequisite knowledge. Sections 12.2, 12.3 in Stewart may be useful as are any number of other resources.

Given two points in the plane corresponding to the vectors $\mathbf{u}_0 = x_0\mathbf{i}+y_0\mathbf{j}$ and $\mathbf{u}_1 = x_1\mathbf{i}+y_1\mathbf{j}$ consider the vector valued function

$$\mathbf{r}(t) = (1-t)\mathbf{u}_0 + t\mathbf{u}_1, \quad 0 \le t \le 1.$$

Give a geometric interpretation of the curve defined by this function. In particular, if a particle's position in the plane was given by $\mathbf{r}(t)$ at each time t, how would that motion be described.

Find two alternative vector valued functions to describe moving in a straight line from the point $(3,7)$ to the point $(-2,4)$.

C. We define the limit

$$\lim_{t\to a}\mathbf{r}(t) = \left(\lim_{t\to a} f(t)\mathbf{i}\right) + \left(\lim_{t\to a} g(t)\mathbf{j}\right)$$

provided these limits exist. With the limit defined component-wise, continuity is similarly defined component-wise. The derivative of the vector valued function $\mathbf{r}$ is defined by

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}.$$

Prove: If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then $\mathbf{r}$ is differentiable at a provided f and g are differentiable at a, and in this case

$$\mathbf{r}'(a) = f'(a)\mathbf{i} + g'(a)\mathbf{j}.$$

Several derivative rules follow from this including

- i $\frac{d}{dt}(\mathbf{r}(t)+\mathbf{u}(t)) = \mathbf{r}'(t) + \mathbf{u}'(t)$
- ii $\frac{d}{dt}(c\mathbf{r}(t)) = c\mathbf{r}'(t)$
- iii $\frac{d}{dt}(f(t)\mathbf{r}(t)) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
- iv $\frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{u}(t)) = \mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t)$
- v $\frac{d}{dt}\mathbf{r}(f(t)) = f'(t)\mathbf{r}'(f(t))$

where $\mathbf{r}$ and $\mathbf{u}$ are differentiable vector valued functions, f is a differentiable scalar valued function, and c is a scalar.

Using defined vector operations and known ordinary derivative rules, **prove** the rules (iii) and (v) above.

D. The derivative $\mathbf{r}'(t)$ vector is tangent to the curve $\mathbf{r}(t)$ at each point for which $\mathbf{r}'(t) \neq 0$. Hence the tangent line to a curve $\mathbf{r}(t)$ at a point P on that curve is the line through P that is parallel to the vector $\mathbf{r}'(t)$ (at that point).

Find a vector valued function for the line tangent to the curve $\mathbf{r}(t) = e^{2t}\mathbf{i}+\ln(t+1)\mathbf{j}$ at the point $(1, 0)$.

Suppose that a particular, differentiable vector valued function $\mathbf{r}$ is always orthogonal to its derivative $\mathbf{r}'$, and that $\mathbf{r}'(t) \neq 0$ for all t. Show that the curve $\mathbf{r}(t)$ is a circle centered at the origin. (**Hint:** Consider $\frac{d}{dt}|\mathbf{r}|^2 = \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t))$.)

E. The definite integral of a vector valued function $\mathbf{r}$ from $t = a$ to $t = b$ is defined as a limit of Riemann sums in the traditional sense

$$\int_a^b \mathbf{r}(t)\, dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\Delta t.$$

And since $\mathbf{r}(t_i^*)\Delta t = f(t_i^*)\Delta t\mathbf{i} + g(t_i^*)\Delta t\mathbf{j}$ we get a component-wise result

$$\int_a^b \mathbf{r}(t)\, dt = \left(\int_a^b f(t)\, dt\right)\mathbf{i} + \left(\int_a^b g(t)\, dt\right)\mathbf{j}.$$

Suppose a particle moves along a path in the plane with velocity given by the vector valued function

$$\mathbf{v}(t) = \left(\frac{4}{1+t^2}\right)\mathbf{i} + \left(\frac{2t}{1+t^2}\right)\mathbf{j}.$$

If the initial position of the particle is the point $\mathbf{p}(0) = \mathbf{i} + 2\mathbf{j}$, determine the position of the particle $\mathbf{p}(t)$ for all t.

Suppose the acceleration $\mathbf{a}(t) = -g\mathbf{j}$ of a particle is due to gravity. Here g is the gravitational constant near the surface of the Earth. Assume that the particle is projected with initial velocity $\mathbf{v}(0) = v_0(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$ from the point $\mathbf{p}(0) = \mathbf{j}$. The value $v_0 > 0$, and α is an acute angle. Determine the trajectory $\mathbf{p}(t)$ of the particle. Assuming that the x-axis is impermeable (the ground, for example), determine the maximum t value for which the vector valued function $\mathbf{p}$ describes the actual motion.

F. Given a curve in the plane, we can ask what its length is. To determine the length, we can consider a partition of the curve given by the points $\{P_0, P_1, \ldots, P_n\}$. Calling a piece of the arc between two partition points Δs, we see that its length is approximated by the hypotenuse of a triangle giving

$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2.$$

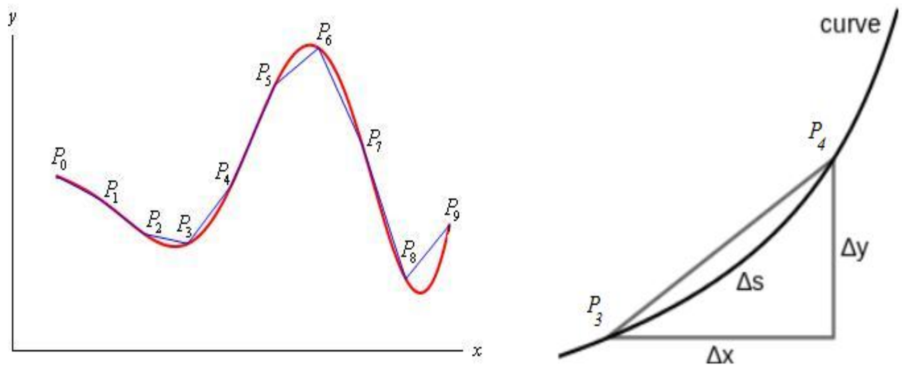

Figure 1: Curve in the plane with a partition. On the right, one subinterval is magnified.

Taking a limit in the usual way and calling the differential arclength parameter ds, we obtain a useful formula

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

If we restrict ourselves to tracing out a curve in the direction of increasing independent variable, we can restate the differential arclength as

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{for} \quad y = f(x)$$

or

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt, \quad \text{for} \quad (x, y) = (f(t), g(t)).$$

In the later case, if the curve is traced out as t increases from a to b, the arclength is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Show that if $\mathbf{r}(t)$ for $a \leq t \leq b$ is differentiable, then the length of the curve is

$$L = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Find the length of half of one arch of the cycloid

$$\mathbf{r}(t) = (t + \sin t)\mathbf{i} + (1 + \cos t)\mathbf{j}, \quad 0 \leq t \leq \pi.$$