MATH 1190 Vocabulary Review

Note: This document was originally written by Dr. Meighan Dillon, and has been edited.

This is a partial list of words and symbols we use frequently in calculus.

The Equal Sign

The equal sign is used in several different ways. In particular, an equation always has an equal sign but some mathematical sentences with equal signs are not equations. You can tell how an equal sign is being used by the context.

1. A *definition*: In, "Let $f(x) = \sin x$," the equal sign indicates the function f is defined by letting $f(x)$ be given by $\sin x$.
2. An *equation*: In, "Solve $x^2 - 1 = 0$," the equal sign indicates something that may be true for some values of x but not other values of x. The values of x for which the equation is true, are *solutions to the equation*.
3. An *identity*: In "$\sin^2 x + \cos^2 x = 1$," the equal sign indicates that no matter what value x has, the sum of the squares of $\sin x$ and $\cos x$ is always 1. This is an equation where every real number is a solution.

Some commonly confused words

1. **Expression** versus **Equation**: An **equation** must have an equal sign, but an **expression** need not. Think of an equation as a **sentence**.
   (a) $x^2 - 1$ is an expression
   (b) $x^2 - 1 = 3$ is an equation
2. **Solve** versus **simplify**: Generally, we **solve** equations, but we **simplify** expressions.
   (a) We can't *solve* the expression $(x + 1)(x - 2)/(2x - 4)$.
   (b) We CAN *simplify* $(x + 1)(x - 2)/(2x - 4)$ (do it!).
3. **Terms** versus **factors**: In a mathematical expression, **terms** are separated by sums or differences. **Factors** are separated by multiplication.
   (a) In $2x^2 + 4xy - 3y$, the terms are $2x^2$, $4xy$, and $3y$.
   (b) The factors in the term $4xy$ above are 4, x, and y, but they are *not* factors in $2x^2 + 4xy - 3y$.

Arithmetic

Arithmetic is adding, subtracting, multiplying and dividing. It can be done with numbers and with functions.

1. Subtracting a is the same as adding $-a$, the **additive inverse** of a.
2. Dividing by b is the same as multiplying by $1/b$, the **multiplicative inverse** of b.
3. **Zero** is special. You can add and subtract zero and you can multiply by zero (that always gets you zero) but you can't divide by it *and get a number.*
   - $1/0 = a$ has *no solutions* (try multiplying both sides by a).
   - $0/0 = a$ could be true for any value of a! We say $0/0$ is thus *indeterminate*. We will study this type of indeterminacy in calculus I. In fact, one of the things you learn quite a bit about in calculus is how to *interpret* division by zero.
4. Multiplying and dividing by -1 *have the effect of changing the sign of an expression.*

Polynomials

Polynomials are expressions we obtain using multiplication and addition on a **variable** x and whatever real numbers we like. For example, $2x$, $3x^2 - \sqrt{2}x + 1/7$ are polynomials *in* x, and $2t + 1$ is a polynomial *in* t.

- The standard form of a polynomial in x is

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

- The a_i's are **coefficients**, a_0 is the **constant term**.
- The highest value of n for which $a_n \neq 0$ is the **degree** of the polynomial.
- For example, $2 - 3x - 7x^3$ has constant term 2 and degree 3; the coefficient of the degree 2 term is zero.
- Constants are considered polynomials of degree 0.
- Degree one polynomials are sometimes called **linear**, as in "$x^2 - 1$ has distinct linear factors, $x - 1$ and $x + 1$."
- Any polynomial with degree greater than 2 can be factored over $\mathbb{R}$.
- The **roots, or zeroes** of a polynomial, $p(x)$, are all numbers a so that $p(a) = 0$.
- Example: 3 is a root of $p(x) = x^2 - 5x + 6$ because $p(3) = 3^2 - 5 \cdot 3 + 6 = 0$.

- To find the roots of a polynomial, $p(x)$, we solve $p(x) = 0$. We **solve polynomial equations by factoring** and using possibly the most important theorem that you learn in high school algebra: the product of numbers is zero if and only if one of the numbers is itself zero. *See D2L for examples of solving polynomial equations.*

- **Roots and factors** are related in an important way: a is a root of a polynomial $p(x)$ if and only if $x - a$ is a factor of $p(x)$.

- The **discriminant** of $ax^2 + bx + c$ is $b^2 - 4ac$. It is important! When $b^2 - 4ac > 0$, $ax^2 + bx + c$ has two distinct real roots, thus distinct linear factors. When $b^2 - 4ac = 0$, $ax^2 + bx + c$ has one real root, thus a single repeated linear factor. When $b^2 - 4ac < 0$, the polynomial has no real roots, and we call it *irreducible*. Those roots are given by the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

More Figuring

- **Taking roots** (not really the same thing as the roots of a polynomial, although related!) is not considered "arithmetic," per se, but is an algebraic activity, meaning that it can be interpreted *in terms of* polynomials. For instance, $\sqrt[3]{2}$ is a root of the polynomial $x^3 - 2$. Note that even roots— square roots, fourth roots, sixth roots— of **negative** numbers are not real, that is, they are complex. For instance, $\sqrt{-2}$ is a root of the polynomial $x^2 + 2$, that is, it is a solution to $x^2 + 2 = 0$, or $x^2 = -2$. But when we square any real number, we get a nonnegative result, so $\sqrt{-2}$ cannot be real. In calculus, we study functions that are real valued so we don't usually think much about things like $\sqrt{-2}$. Understand the phrase "factor over $\mathbb{R}$" to mean: factor so that no numbers you write down are complex, all are real.

- **Infinity** is not a number. The word itself means "beyond number." Think of ∞ as bigger than any number and that will serve you well. Though ∞ is not a number, there is a certain amount of arithmetic that we can do meaningfully with it and that is one of the things we learn in calculus. For example, $\infty + \infty = \infty$. Also, if a is any real number $a + \infty = \infty$ and if $a > 0$, $a \cdot \infty = \infty$. The list of things we *cannot* do with infinity (and zero!) is a list of *indeterminate forms*, like $0/0$. Another indeterminate is $0 \cdot \infty$. We will discuss indeterminate forms more this semester.

Functions

Functions are an important device in calculus. For us, a *function* is a **rule** defined to assign a single real number value to each element in a certain set of real numbers called its **domain**. If x belongs to the domain, then f assigns to x *exactly one* real number, which may be designated $f(x)$. The set of all $f(x)$ for x is in the domain is the **range** of f. The **graph**

of f is the set of all points $(x, f(x))$ in the xy-plane, with x in the domain. To emphasize the relationship between the graph of f and f itself, we often write $f(x) = y$.

- **Polynomial functions** have domain all of $\mathbb{R}$. Non-vertical lines are graphs of functions $f(x) = ax + b$, that is, degree one polynomial functions in x. The graph of a degree two polynomial function in x is a parabola.

  Often when speaking of "polynomial functions" in calculus, we just say "polynomial." This is one of many things that are shortened through usage. Another example is the concept of "area of a circle" which is short for "area enclosed by a circle."

- **Rational functions** are those that can be written $p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials. Polynomials form a subclass of the rational functions. For example $f(x) = 1/x$, $g(x) = \dfrac{2x+3}{5x-1}$, and $h(x) = 2x^2 + 3$ are all rational functions. A rational function $f(x) = p(x)/q(x)$ has as its domian the set of all x so that $q(x) \neq 0$: we don't get a number when we divide by zero and the output for a function, in calculus, is always a real number.

- **Algebraic functions** include the rational functions and functions that involve taking roots. For example, $f(x) = \sqrt{x}$ is algebraic, as is $g(x) = \sqrt[3]{\dfrac{x+1}{x+3}}$ and neither of these is rational. Note that if y is given by an algebraic function in x, then there is a polynomial equation relating x and y. For example, if $y = \sqrt[3]{\dfrac{x+1}{x+3}}$, then $y^3 = \dfrac{x+1}{x+3}$ so $xy^3 + 3y^3 = x + 1$.

  - The *domain* of an algebraic function in x is the set of all x that won't force you to divide by zero or to take an even root of a negative number.
  - A hint for simplifying algebraic functions: Try mulitplying by the *conjugate* of the denominator. For example, if the denominator is $2 + \sqrt{x}$, try multiplying the top and bottom of the expression by $2 - \sqrt{x}$. *See D2L for an example.*

- **Transcendental functions** cannot be defined using polynomials. The trigonometric and inverse trigonometric functions are transcendental, as are logarithms and exponentials. We will work extensively with transcendental functions this semester.

- **Piecewise functions** are defined in different ways on different pieces of their domains. These are not algebraic but usually consist of pieces of algebraic functions and maybe some other functions. They give us important examples in calculus so you have to get used to them. In order to graph them, graph each function individually on the specified domains. *See examples in D2L.*

- **Operations with functions** include arithmetic and **function composition**. For example, $h(x) = \sin(x^2)$ is the composition $(f \circ g)(x)$, where $g(x) = x^2$ and $f(x) = \sin x$.

Composition is the most difficult operation with functions because you don't do it with numbers, so you aren't as familiar with it.

Graphs of functions

The **graph of a function** $y = f(x)$ is a plot in the plane of all points (x, y) satisfying the relationship $y = f(x)$. For example, since $3^2 = 9$, the point $(3, 9)$ appears on the graph of the function $y = x^2$, as does $(-\sqrt{5}, 5)$.

- The **x-intercepts** of the graph of $y = f(x)$ are the points at which the graph intersects the x axis. Since the y-coordinate at the x axis is 0, the x-intercepts are found by solving the equation $f(x) = 0$. Note that the process by which you find the x-intercepts of a polynomial function is the same as that for finding the roots - that is, the roots of the polynomial *are* the x-coordinates of the x-intercepts.

- The **y-intercept** of the graph of $y = f(x)$ is the point at which the graph intersects the y axis. Since the x-coordinate at the y axis is 0, the y-intercept is found by evaluating $f(0)$.

- Note that if f is a function, there will be at most one y intercept, whereas there may be infinitely many x-intercepts (as in the function $f(x) = \sin x$).

- It can be very helpful to recognize when the graph of a function has symmetry.

  1. **Even functions** are symmetric with respect to the y-axis; that is, if you fold the function over the y-axis, the positive x-region and the negative x-region will line up. Even functions satisfy the equation $f(x) = f(-x)$ for every x. The example to keep in mind is $f(x) = x^2$.
  2. **Odd functions** are symmetric with respect to the origin; the example to think of here is $f(x) = x^3$. Odd functions satisfy the equation $f(x) = -f(-x)$ for every x.
  3. No function can be symmetric with respect to the x-axis, because then it wouldn't be a function!
  4. See D2L for examples of how to test functions for symmetry.

Triangles

- **The Pythagorean Theorem** is true only of right triangles: the sum of the squares of the legs is the square of the hypotenuse. This simple fact comes up surprisingly often, so never forget it!

- **Similar triangles** have the same shape at different scales.

  - Similar triangles have the same angles and corresponding legs have a fixed proportion.

- For example, triangles with legs 3-4-5 and 6-8-10 are similar. All **equilateral triangles** (all three legs with the same length) are similar.
- An important trick to remember for some applied problems you will do this semester: If you have a triangle sitting in front of you, and draw a line segment inside the triangle parallel to one side, the smaller triangle you just made is similar to the original.
- Visually, this looks like "nested triangles", and they are always similar.

- **Isosceles triangles** have exactly two legs with the same length. Most isosceles triangles are not right triangles - only those with two 45 degree angles are.
- **Triangle area** is the area of the region bounded by a triangle. It is given by $1/2bh$, where b is the length of *any* side of the triangle and h is the distance from that side to the opposite vertex.

Other shapes

- **Circles**: the area enclosed by a circle is πr^2, where r is the radius, and the circumference is $2\pi r$. The equation of a circle in the x, y plane centered at the origin is $x^2 + y^2 = r^2$.
- **Rectangles:** Have area lw, where l is the length and w is the width
- **Rectangular prisms:** Have volume lwh, where l,w, and h are the lengths of its three dimensions. The *don't* have area because they are 3-dimensional, but you can find the **surface area** by adding the area of each of the six sides.
- **Angles:** We always measure angles in **radians** in calculus. To convert between radians and degrees, use $180° = \pi$. You should be able to do conversions of commn angles (like $\pi/6$ very quickly.