Exam I Math 2253H sec. 5H

Fall 2014

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems. The point value for each problem is listed with the problem. There are no notes, or books allowed and **no calculator is allowed.** To receive full credit, you must clearly justify your answer and use proper notation.

(1) Evaluate the limit. (10 points)

$$\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} = \lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} \cdot \frac{4+\sqrt{x}}{4+\sqrt{x}}$$

$$= \lim_{x\to 16} \frac{16-x}{(x-16)(4+\sqrt{x})}$$

$$= \lim_{x\to 16} \frac{-1}{4+\sqrt{x}} = \frac{-1}{4+\sqrt{16}} = \frac{-1}{8}$$

(2) Evaluate the limit. (10 points)

$$\lim_{v\to 4^+} \frac{v-4}{|4-v|}$$

$|4-v| = -(4-v)$ for $4-v<0$

i.e. $v>4$

$$= \lim_{v\to 4^+} \frac{v-4}{-(4-v)}$$

$$= \lim_{v\to 4^+} \frac{v-4}{v-4} = 1$$

(3) Suppose that a particle moves along the x-axis so that its position s in feet at time t in seconds is

$$s = f(t) = \sqrt{2t}, \quad \text{for } t \geq 0.$$

(a) Find the average velocity of the particle over the interval from $t = 0$ to $t = 2$ seconds. (5 points)

$$\text{avg. vel} = \frac{f(2)-f(0)}{2-0} = \frac{\sqrt{4}-\sqrt{0}}{2-0} = 1 \ \frac{ft}{sec}.$$

(b) Find the instantaneous velocity[1] of the particle when $t = 2$ seconds. (10 points)

$$v(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2(2+h)} - \sqrt{2\cdot 2}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4+2h} - 2}{h} \cdot \frac{\sqrt{4+2h}+2}{\sqrt{4+2h}+2}$$

$$= \lim_{h\to 0} \frac{4+2h-4}{h(\sqrt{4+2h}+2)}$$

$$= \lim_{h\to 0} \frac{2h}{h(\sqrt{4+2h}+2)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{4+2h}+2} = \frac{2}{\sqrt{4}+2} = \frac{1}{2}$$

$$v(2) = \frac{1}{2} \ \frac{ft}{sec}.$$

[1] To receive credit, use the definition of instantaneous velocity—i.e. set up and evaluate a limit. Use of *derivative rules* will not receive credit.

(4) For each limit statement, determine a choice of δ (in terms of ϵ) that can be used to formally prove[2] the truth of the limit statement. (10 points each)

$f(x) = 2x+5, \quad a = -1, \quad L = 3$

(a) $\lim_{x \to -1} 2x+5 = 3$

We need $|f(x) - L| < \varepsilon$ when $|x - a| < \delta$

$|2x+5-3| = |2x+2| = 2|x+1|$

If $|x+1| < \delta$ then $2|x+1| < 2\delta$

So $2|x+1| < \varepsilon$ if $\varepsilon = 2\delta \implies \delta = \frac{\varepsilon}{2}$

We can take $\delta = \frac{\varepsilon}{2}$.

(b) $\lim_{x \to 1} \frac{x+3}{4} = 1$ $\quad f(x) = \frac{x+3}{4}, \quad a = 1, \quad L = 1$

$|f(x) - L| = \left|\frac{x+3}{4} - 1\right| = \left|\frac{1}{4}(x+3-4)\right| = \frac{1}{4}|x-1|$

we need $\frac{1}{4}|x-1| < \varepsilon$.

If $|x-1| < \delta$ then $\frac{1}{4}|x-1| < \frac{1}{4}\delta$

So choose $\frac{1}{4}\delta = \varepsilon \implies \delta = 4\varepsilon$

We can set $\delta = 4\varepsilon$.

[2]Note: It is not necessary to produce a proof. The preliminary scratch work, and what you can conclude from the scratch work, is sufficient. Do show the work though. We didn't prove the conjecture from class.

(5) Let f be defined as follows: (here a and b are constants)

$$f(x) = \begin{cases} x-1, & x < -1 \\ ax, & -1 \le x < 2 \\ b - x, & x \ge 2 \end{cases}$$

(a) Evaluate (5 points)

$\lim_{x \to -1^-} f(x) =$ -2 and $f(-1) =$ $-a$

(b) Determine the value of a such that f is continuous at $x = -1$. (10 points)

to have $\lim_{x \to -1} f(x) = f(-1)$ set $-2 = -a$

$\Rightarrow \quad a = 2.$

(c) For the value of a found in part (b), evaluate (5 points)

$\lim_{x \to 2^-} f(x) =$ $2a = 4$ and $f(2) =$ $b-2$

$*$ $a=2$

(d) Determine the value of b such that f is continuous at $x = 2$. (10 points)

To have $\lim_{x \to 2} f(x) = f(2)$ we need

$4 = b - 2 \quad \Rightarrow \quad b = 6.$

f will be continuous @ -1 and 2 when $a = 2$ and $b = 6$.

(6) It can be shown that $\cos\theta \leq \dfrac{\sin\theta}{\theta} \leq 1$ for all $\theta \neq 0$ in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Evaluate $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$ (10 points)

$\displaystyle\lim_{\theta\to 0} \cos\theta = \cos(0) = 1$

and

$\displaystyle\lim_{\theta\to 0} 1 = 1$

By the squeeze theorem

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

(7) Use the Intermediate Value Theorem to argue that the equation has at least one solution in the given interval. (5 points)

$$\sqrt{x+1} = x^2 \quad \text{in} \quad [0, 3]$$

Set $f(x) = \sqrt{x+1} - x^2$. f is continuous for all $x \geq -1$.

$f(0) = \sqrt{1} - 0 = 1$

$f(3) = \sqrt{4} - 3^2 = -5$

$f(0) > 0$ and $f(3) < 0$

So zero is a number between $f(0)$ and $f(3)$.

By the IVT, there exists some c in $(0,3)$ such that $f(c) = 0$.

$\Rightarrow \sqrt{c+1} - c^2 = 0$

$\sqrt{c+1} = c^2$.