Exam I Math 2335 sec. 51

Spring 2016

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use your text book (Atkinson & Han), one 8.5" by 11" page of written notes and a calculator. To use an e-version of the book, you must disable your wifi or 4G (3G) access. **NO use of internet access devices is permited. Illicit use of such items will result in a grade of zero on this exam and a formal allegation of academic misconduct.** To receive full credit, you must follow the directions given and clearly justify your answer.

(1) The sine cardinal function $\operatorname{sinc}(x)$ is defined by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

The Taylor polynomial with remainder for $\operatorname{sinc}(x)$ is

$$\operatorname{sinc}(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + (-1)^{n+1} \frac{x^{2n+2}}{(2n+3)!} \cos(c)$$

where c is some number between zero and x.

Find a bound on the error $|\operatorname{sinc}(x) - p_4(x)|$ when p_4 is used to approximate $\operatorname{sinc}(x)$ for $0 \leq x \leq 1$. You may give your answer as an exact rational number or as a decimal with 6 digits to the right of the decimal place. Do **not** give your answer in scientific notation.

If $2n = 4$, then $n = 2$. Hence

$$R_4(x) = (-1)^3 \frac{x^{4+2}}{(4+3)!} \operatorname{Cos}(c)$$ for some c between 0 and x

$$= \frac{-x^6}{7!} \operatorname{Cos}(c)$$

Note that if $0 \leq x \leq 1$, we have $0 \leq c \leq 1$

$|x|^6 \leq 1$ and $|\operatorname{Cos}(c)| \leq 1$

Hence

$$|\operatorname{Sinc}(x) - p_4(x)| = |R_4(x)| = \left| \frac{-x^6}{7!} \operatorname{Cos} c \right|$$

$$\leq \frac{1}{7!} \cdot 1 = \frac{1}{5040}$$

$$\doteq 0.000198$$

for $0 \leq x \leq 1$

(2) The number $x_A = 3.433$ is an approximation to the value x_T that is correctly rounded to the digits shown.

(a) Determine the largest interval that contains x_T.

$$3.4325 \le x_T < 3.4335$$

(b) Bound the error $\mathrm{Err}(x_A)$.

$$|x_T - x_A| \le 0.0005$$

(c) Use the Mean value theorem* to bound the propagated error when $\ln(x_A)$ is used to approximate $\ln(x_T)$. Give 6 digits to the right of the decimal; do **not** give your answer in scientific notation.

For $f(x) = \ln x$, $f'(x) = \frac{1}{x}$

Since $3.4325 \le x_T < 3.4335$, If c is between x_T and x_A we know that

$$3.4325 \le c < 3.4335$$

Thus $|f'(c)| \le \frac{1}{3.4325}$.

We have

$$|\ln(x_T) - \ln(x_A)| = |f'(c)(x_T - x_A)|$$

$$\le \frac{1}{3.4325} \cdot 0.0005 \doteq 0.000146$$

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*$f(b) - f(a) = f'(c)(b-a)$ for some c between a and b

(3) Suppose we wish to determine the value of $\sqrt[3]{2}$.

(a) Let $f(x) = x^3 - 2$ and write out the Newton iteration formula. Simplify to a single rational expression.

$$f'(x) = 3x^2, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2}$$

$$= x_n - \frac{1}{3}x_n + \frac{2}{3x_n^2} = \frac{2}{3}x_n + \frac{2}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2}}$$

(b) Taking an initial guess of $x_0 = 1.00000$, compute the next four iterations of Newton's method. Fill in the table giving 6 digits to the right of the decimal.

n	x_n
0	1.000000
1	1.333333
2	1.263889
3	1.259933
4	1.259921

(c) Determine the number of iterations that would be required to find $\sqrt[3]{2}$ by the bisection method with an initial interval of $[a, b] = [1, 2]$ correct to within an error of $\epsilon = 10^{-9}$.

$$n \geq \frac{\ln\left(\frac{2-1}{10^{-9}}\right)}{\ln 2} \doteq 29.9$$

Hence 30 iterations are required.

(4) Suppose that we have the following information about a function $f(x)$.

$$f(0) = 3, \quad f'(0) = 2, \quad f''(0) = 2, \quad f'''(0) = -3, \quad \text{and} \quad f^{(4)}(x) = \frac{2x+8}{(x+1)^4}$$

Write out the Taylor polynomial $p_3(x)$ of degree 3 and the remainder $R_3(x)$ for f centered at $a = 0$.

$$P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$P_3(x) = 3 + 2x + x^2 - \frac{1}{2}x^3$$

$$R_3(x) = \frac{x^4}{4!}\left(\frac{2c+8}{(c+1)^4}\right)$$ for some c between x and zero.

(5) Using Taylor polynomials, algebra or function identities, find an alternative expression that will avoid loss of signficance errors when evaluated near the indicated x.

(a) $f(x) = \sqrt{x^2+x} - x$, x very large

$$f(x) = (\sqrt{x^2+x} - x)\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \frac{x}{\sqrt{x^2+x}+x}$$

(b) $g(x) = \dfrac{\text{sinc}\,(x) - 1}{x}$ $\quad x \approx 0$ (Hint: see problem 1.)

From problem #1

$$g(x) = \frac{-x}{3!} + \frac{x^3}{5!} + \dots + (-1)^n \frac{x^{2n-1}}{(2n+1)!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+3)!}\cos c$$

for some c between zero and x

(6) The cubic equation $x^3 - x - 1 = 0$ has one real root.

(a) Write out the Secant method iteration formula to find this root. (Simplify within reason.)

$f(x) = x^3 - x - 1$

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = x_n - (x_n^3 - x_n - 1)\frac{x_n - x_{n-1}}{x_n^3 - x_n - x_{n-1}^3 + x_n}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - x_n - 1)(x_n - x_{n-1})}{x_n^3 - x_n - x_{n-1}^3 + x_{n-1}}$$

(b) Taking $x_0 = 1$ and $x_1 = 2$, compute the next two iterates x_2 and x_3 obtained using the Secant method. Give your answers as either exact rational numbers or decimal numbers with 6 digits to the right of the decimal.

$x_2 = 2 - \dfrac{(2^3 - 2 - 1)(2 - 1)}{2^3 - 2 - 1^3 + 1} = \dfrac{7}{6} \doteq 1.166667$

$x_3 = x_2 - \dfrac{(x_2^3 - x_2 - 1)(x_2 - 2)}{x_2^3 - x_2 - 2^3 + 2} = \dfrac{302}{241}$

$\doteq 1.253112$