Exam II Math 2335 sec. 51

Spring 2016

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use your text book (Atkinson & Han), one 8.5" by 11" page of written notes and a calculator. To use an e-version of the book, you must disable your wifi or 4G (3G) access. **NO use of internet access devices is permited. Illicit use of such items will result in a grade of zero on this exam and a formal allegation of academic misconduct.** To receive full credit, you must follow the directions given and clearly justify your answer.

(1) Determine all possible values of the numbers A and B such that the given function is a cubic spline on the interval $[1, 3]$. (Hint: Consider the continuity conditions for a cubic spline.)

$$s(x) = \begin{cases} x^3 - 3x^2 + 2x + A, & 1 \le x \le 2 \\ -x^3 + 9x^2 + Bx + 4, & 2 \le x \le 3 \end{cases}$$

By continuity of $s(x)$

$$\lim_{x \to 2^+} s(x) = \lim_{x \to 2^-} s(x)$$

So
$$2^3 - 3 \cdot 2^2 + 2 \cdot 2 + A = -2^3 + 9 \cdot 2^2 + B(2) + 4$$
$$8 - 12 + 4 + A = -8 + 36 + 2B + 4$$
$$A = 32 + 2B$$

By continuity of $s'(x)$

$$\lim_{x \to 2^+} s'(x) = \lim_{x \to 2^-} s'(x)$$

$$s'(x) = \begin{cases} 3x^2 - 6x + 2, & 1 \le x \le 2 \\ -3x^2 + 18x + B, & 2 \le x \le 3 \end{cases}$$

$$3 \cdot 2^2 - 6 \cdot 2 + 2 = -3 \cdot 2^2 + 18 \cdot 2 + B$$
$$12 - 12 + 2 = -12 + 36 + B \Rightarrow B = 2 - 24 = -22$$

Then $A = 32 + 2B = 32 - 44 = -12$

There is one solution $A = -12$, $B = -22$.

(2) Consider the integral $\int_0^4 \sqrt{1+x^2}\,dx$

(a) Approximate the integral with the trapezoid rule* $T_4(f)$. Give 5 digits to the right of the decimal. Do not give your answer in scientific notation.

$h = \frac{4-0}{4} = 1 \qquad x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$

$f(0) = \sqrt{1} = 1,\ f(1) = \sqrt{2},\ f(2) = \sqrt{5},\ f(3) = \sqrt{10},\ f(4) = \sqrt{17}$

$T_4(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$

$= \frac{1}{2}\left[1 + 2\sqrt{2} + 2\sqrt{5} + 2\sqrt{10} + \sqrt{17}\right]$

$\doteq 9.37411$

(b) Approximate the integral with Simpson's rule $S_4(f)$. Give 5 digits to the right of the decimal. Do not give your answer in scientific notation.

The same h, x_i, and function values are used.

$S_4 = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$

$= \frac{1}{3}\left[1 + 4\sqrt{2} + 2\sqrt{5} + 4\sqrt{10} + \sqrt{17}\right]$

$\doteq 9.30040$

*To receive credit, the set up details must be shown for both (a) and (b).

(3) Suppose we wish to interpolate f with $P_3(x)$ on the interval $[-1,1]$ where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

(a) It can be shown that $|f^{(4)}(x)| \leq \frac{1}{5}$ when $-1 \leq x \leq 1$. Find a bound on the error $|f(x) - P_3(x)|$ if the optimal error reducing nodes are used. Express the bound exactly or give 5 decimal digits.

For the Chebyshev nodes

$$|f(x) - P_3(x)| \leq \frac{L}{2^3} \quad \text{where} \quad L = \max_{[-1,1]} \left| \frac{f^{(4)}(x)}{4!} \right|$$

Given $|f^{(4)}(x)| \leq \frac{1}{5}$, $L = \frac{1/5}{4!} = \frac{1}{5 \cdot 24} = \frac{1}{120}$

So

$$|f(x) - P_3(x)| \leq \frac{\frac{1}{120}}{8} = \frac{1}{960} \doteq 0.00104$$

(b) Suppose instead that equally spaced nodes are used. Find a bound on the error in this case†. (Again, exact or with 5 decimal digits.)

For equally spaced nodes $h = \frac{1-(-1)}{3} = \frac{2}{3}$

and

$$|f(x) - P_3(x)| = \left| \Psi_3(x) \frac{f^{(4)}(c)}{4!} \right| \quad \text{for some } c \text{ in } [-1,1].$$

$$|\Psi_3(x)| \leq h^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81} \quad \text{and} \quad \frac{|f^{(4)}(c)|}{4!} \leq \frac{1}{120} \text{ from (a)}$$

So

$$|f(x) - P_3(x)| \leq \frac{16}{81} \cdot \frac{1}{120} = \frac{2}{1215} \doteq 0.00165$$

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†Recall $|\Psi_3(x)| \leq h^4$ for equally spaced nodes h units apart.

(4) Consider the data given in the table.

x	0.1	0.2	0.3
$f(x)$	-0.23	-0.32	-0.36

Find $P_2(x)$ using the Newton divided differences formulation.

$$f[0.1, 0.2] = \frac{-0.32 - (-0.23)}{0.2 - 0.1} = 10(-0.09) = -0.9$$

$$f[0.2, 0.3] = \frac{-0.36 - (-0.32)}{0.3 - 0.2} = 10(-0.04) = -0.4$$

so

$$f[0.1, 0.2, 0.3] = \frac{f[0.2, 0.3] - f[0.1, 0.2]}{0.3 - 0.1}$$

$$= 5(-0.4 - (-0.9)) = 2.5$$

$$P_2(x) = f(0.1) + f[0.1, 0.2](x - 0.1) + f[0.1, 0.2, 0.3](x - 0.1)(x - 0.2)$$

$$\boxed{P_2(x) = -0.23 - 0.9(x - 0.1) + 2.5(x - 0.1)(x - 0.2)}$$

(5) The integral $I(f) = \int_a^b f(x)\,dx$ was approximated using the trapezoid and Simpson's rule for several choices of equally spaced partitions. The results were recorded in the following tables.

$T_2(f)$	0.697996
$T_4(f)$	0.704834
$T_8(f)$	0.706539

$S_2(f)$	0.707201
$S_4(f)$	0.707113
$S_8(f)$	0.707107

(a) Compute the Richardson Extrapolation approximation R_4 for the Trapezoid rule.

$$R_4 = \frac{1}{3}(4T_4 - T_2)$$

$$= \frac{1}{3}(4 \cdot 0.704834 - 0.697996)$$

$$\doteq 0.707113$$

(b) Compute the Richardson Extrapolation approximation R_4 for Simpson's rule.

$$R_4 = \frac{1}{15}(16 S_4 - S_2)$$

$$= \frac{1}{15}(16 \cdot 0.707113 - 0.702201)$$

$$\doteq 0.707107$$

(c) The Midpoint rule can be defined as $M_n = \frac{1}{2}(3S_{2n} - T_n)$. Compute M_4.

IF $n=4$, then $2n=8$.

$$M_4 = \frac{1}{2}(3S_8 - T_4)$$

$$= \frac{1}{2}(3 \cdot 0.707107 - 0.704834)$$

$$\doteq 0.708244$$

(6) Suppose we wish to approximate the improper (but convergent) integral $\int_{-1}^{1} \frac{1}{\sqrt[4]{1-x^2}}\,dx$.

(a) Explain clearly (and in plain English) why it is not possible to compute S_n for any even number $n = 2, 4, \ldots$ (Hint: Consider why the integral is improper.)

For any n, $x_0 = -1$ and $x_n = 1$. With

$f(x) = \frac{1}{\sqrt[4]{1-x^2}}$, neither $f(1)$ nor $f(-1)$

are defined. Hence S_n can not be

computed as it requires these values.

(Code written to do this with crash.)

(b) Evaluate the integral using Gaussian quadrature $I_2(f)$. Give your answer with 5 digits to the right of the decimal.

$w_1 = w_2 = 1$ and $-x_1 = x_2 = \frac{1}{\sqrt{3}}$

with $f(x) = \frac{1}{\sqrt[4]{1-x^2}}$

$$I_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt[4]{1-\left(\frac{-1}{\sqrt{3}}\right)^2}} + \frac{1}{\sqrt[4]{1-\left(\frac{1}{\sqrt{3}}\right)^2}}$$

$$= \frac{1}{\sqrt[4]{2/3}} + \frac{1}{\sqrt[4]{2/3}} = 2\left(\frac{3}{2}\right)^{1/4} \doteq 2.21336$$