Final Exam Math 2253 sec. 4

Summer 2011

Name: (1 point) Solutions

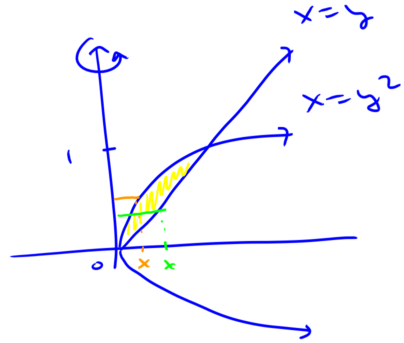
Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

INSTRUCTIONS: There are 10 problems worth 11 points each. You can eliminate any one problem, or I will count your best 9 out of 10. There are no notes, or books allowed and **no calculator is allowed.** To receive full credit, you must clearly justify your answer.

(1) The region bounded by the curves $x = y^2$ and $x = y$ is rotated about the y-axis. Find the volume of the resulting solid.



Intersections:

$$x = y^2 = y$$
$$y^2 - y = 0$$
$$y(y-1) = 0$$
$$y = 0 \text{ or } y = 1$$

Outer radius: $R = x = y$ to the line

Inner radius: $r = x = y^2$ to the parabola

for some y in $(0,1)$ one washer has volume

$$V = (\pi R^2 - \pi r^2)\,\Delta y = \pi(y^2 - y^4)\,\Delta y$$

The total volume is

$$V = \int_0^1 \pi(y^2 - y^4)\,dy$$
$$= \pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right|_0^1 = \pi\left[\frac{1}{3} - \frac{1}{5} - 0\right] = \frac{2\pi}{15}$$

(2) Evaluate the limit

$$\lim_{x\to 5} \frac{x-5}{\sqrt{x^2-9}-4} = \lim_{x\to 5} \frac{x-5}{\sqrt{x^2-9}-4}\left(\frac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4}\right)$$

$$= \lim_{x\to 5} \frac{(x-5)(\sqrt{x^2-9}+4)}{x^2-9-16}$$

$$= \lim_{x\to 5} \frac{(x-5)(\sqrt{x^2-9}+4)}{x^2-25}$$

$$= \lim_{x\to 5} \frac{(x-5)(\sqrt{x^2-9}+4)}{(x-5)(x+5)}$$

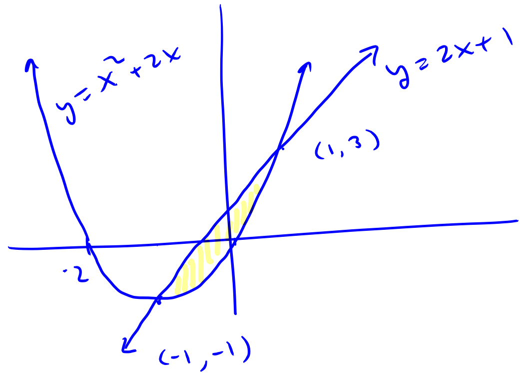
$$= \lim_{x\to 5} \frac{\sqrt{x^2-9}+4}{x+5} = \frac{\sqrt{16}+4}{5+5} = \frac{4}{5}$$

(3) Find the derivative of the function. Simplification is NOT necessary.

$$g(t) = \frac{t^{5/2}-3t}{4t^3+\tan t}$$

$$g'(t) = \frac{\left(\frac{5}{2}t^{3/2}-3\right)(4t^3+\tan t)-(t^{5/2}-3t)(12t^2+\sec^2 t)}{(4t^3+\tan t)^2}$$

(4) Find the area of the region bounded by the curves $y = x^2 + 2x$ and $y = 2x + 1$.



Intersections:

$x^2+2x = 2x+1$

$x^2 = 1 \quad x = 1$ or $x = -1$

Top: line $y = 2x+1$

bottom: parabola $y = x^2+2x$

$$A = \int_{-1}^{1} \left(2x+1 - (x^2+2x)\right) dx = \int_{-1}^{1} (1-x^2)\, dx = x - \frac{x^3}{3} \Big|_{-1}^{1}$$

$$= \left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right) = 2 - \frac{2}{3} = \frac{4}{3}$$

(5) Evaluate the limit.

$$\lim_{x\to\infty} \frac{4x - 2x^2}{x^2 + 7x + 1} = \lim_{x\to\infty} \frac{4x - 2x^2}{x^2+7x+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{4}{x} - 2}{1 + \frac{7}{x} + \frac{1}{x^2}} = \frac{0-2}{1+0} = -2$$

(6) Find the equation of the line tangent to the graph of $x^2y + y^2x = 2$ at the point $(1, 1)$.

Find $\frac{dy}{dx}$: $2xy + x^2\frac{dy}{dx} + 2yx\frac{dy}{dx} + y^2 = 0$

$$(x^2+2xy)\frac{dy}{dx} = -(2xy+y^2)$$

$$\frac{dy}{dx} = \frac{-(2xy+y^2)}{x^2+2xy}$$

Slope $m = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = -\frac{3}{3} = -1$

$$y-1 = -1(x-1)$$

$$\boxed{y = -x+2}$$

(7) Evaluate the definite integral.

$$\int_0^1 x\sqrt{x^2+1}\,dx$$

Let $u = x^2+1$, $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

If $x=0$, $u=1$

$x=1$, $u=2$

$$= \int_1^2 \frac{1}{2}u^{1/2}\,du$$

$$= \frac{1}{2}\left.\frac{u^{3/2}}{3/2}\right|_1^2 = \left.\frac{1}{3}u^{3/2}\right|_1^2$$

$$= \frac{1}{3}(2)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{2\sqrt{2}}{3} - \frac{1}{3}$$

(8) Solve the initial value problem.

$$\frac{dy}{dx} = 4x + \sec x \tan x, \quad y(0) = -2$$

$$y = \int (4x + \sec x \tan x)\,dx$$

$$= 4\frac{x^2}{2} + \sec x + C$$

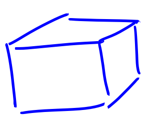
$$y(x) = 2x^2 + \sec x + C$$

$$y(0) = 2 \cdot 0 + \sec 0 + C = -2$$

$$1 + C = -2 \implies C = -3$$

$$\boxed{y = 2x^2 + \sec x - 3}$$

(9) The volume of a cube is increasing at a rate of 1200 cm^3/min at the instant that its edges are 20 cm long? At what rate are the lengths of the edges changing at that instant?



|- x -| cm

Let x be the edge length. $V = x^3$

Given $\frac{dV}{dt} = 1200 \frac{cm^3}{min}$ when $x = 20$ cm.

Find $\frac{dx}{dt}$ when $x = 20$ cm.

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \implies 1200 \frac{cm^3}{min} = 3(20\ cm)^2 \frac{dx}{dt} \quad \text{when } x = 20\ cm$$

$$\frac{dx}{dt} = \frac{1200 \frac{cm^3}{min}}{3 \cdot 400\ cm^2} = 1 \frac{cm}{min}$$

They are increasing at the rate of $1 \frac{cm}{min}$ at that moment.

(10) Evaluate the indefinite integral.

(a) $\displaystyle\int \cot^4 x \csc^2 x\, dx$

Let $u = \cot x$ $\quad du = -\csc^2 x\, dx$

$-du = \csc^2 x\, dx$

$= -\int u^4\, du$

$= -\frac{u^5}{5} + C$

$= -\frac{\cot^5 x}{5} + C$

(b) $\displaystyle\int \frac{5r}{(4+r^2)^2}\, dr$

Let $u = 4 + r^2$, $\quad du = 2r\, dr$

$\frac{1}{2} du = r\, dr$

$= \int \frac{5}{u^2} \cdot \frac{1}{2}\, du$

$= \frac{5}{2} \int u^{-2}\, du = \frac{5}{2} \frac{u^{-1}}{-1} + C$

$= \frac{-5}{2u} + C = \frac{-5}{2(4+r^2)} + C$