Final Exam Math 2253H sec. 5H

Fall 2014

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

INSTRUCTIONS: There are 11 problems worth 10 points each. You may eliminate any one problem, or I will drop the problem with your lowest score. There are no notes, or books allowed and **no calculator is allowed. Illicit use of a calculator, smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam and may result in a formal reported allegation of academic misconduct.** To receive full credit, you must clearly justify your answer and use proper notation.

(1) Consider the function $f(x) = \begin{cases} \frac{\sqrt{x^2+3}-2}{x-1}, & x \neq 1 \\ c, & x = 1. \end{cases}$

(a) Evaluate $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1} \cdot \left(\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} \right)$

$$= \lim_{x\to 1} \frac{x^2+3-4}{(x-1)(\sqrt{x^2+3}+2)} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+3}+2)}$$

$$= \lim_{x\to 1} \frac{x+1}{\sqrt{x^2+3}+2} = \frac{2}{4} = \frac{1}{2}$$

(b) Determine the value of c such that f is continuous at one.

$$\lim_{x\to 1} f(x) = f(1) \Rightarrow c = \frac{1}{2}$$

(2) Evaluate each limit. (Use of l'Hopital's rule will not be considered for credit.)

(a) $\lim_{x \to 0} \frac{\tan(3x)}{x}$ $= 3 \lim_{x \to 0} \frac{\tan(3x)}{3x} = 3 \cdot 1 = 3$

(b) $\lim_{t \to \infty} \frac{\sqrt{4t^2+1}}{t-2}$ $\cdot \frac{\frac{1}{t}}{\frac{1}{t}} = \lim_{t \to \infty} \frac{\sqrt{4 + 1/t^2}}{1 - \frac{2}{t}} = \frac{\sqrt{4}}{1} = 2$

(c) $\lim_{x \to -2} \frac{x^2-4}{x+2}$ $= \lim_{x \to -2} \frac{(x-2)(x+2)}{x+2}$

$= \lim_{x \to -2} x - 2 = -4$

(3) Find $\frac{dy}{dx}$. Do not leave compound fractions in your answers.

(a) $y = \dfrac{x^2+4}{x+3}$

$$y' = \frac{2x(x+3) - (x^2+4)}{(x+3)^2}$$

$$= \frac{x^2+6x-4}{(x+3)^2}$$

(b) $y = \displaystyle\int_1^{x^3} \frac{dt}{t^3+1}$

$$y' = \frac{3x^2}{x^9+1}$$

(c) $x^2+2xy-y^3 = 1$

$$2x + 2y + 2x\,y' - 3y^2\,y' = 0$$

$$(2x+2y) = (3y^2-2x)\,y'$$

$$\Rightarrow \quad y' = \frac{2x+2y}{3y^2-2x}$$

(4) Find $\frac{dy}{dx}$. Do not leave compound fractions in your answers.

(a) $y = \cos(\sin x)$

$y' = -\sin(\sin x)\cos x$

(b) $y = \sin(\cos x)$

$y' = \cos(\cos x)(-\sin x)$

$= -\cos(\cos x)\sin x$

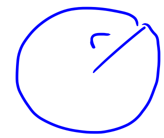
(c) $y = \cot(2x)$

$y' = -\csc^2(2x) \cdot 2$

$= -2\csc^2(2x)$

(5) Oil drips from a car leaving a circular puddle. At the moment that the area of the circular puddle is 36π square inches, its area is increasing at a rate of $1/2$ square inches per minute. Determine the rate at which the radius is increasing at this moment. (Your answer should include appropriate units.)

$A = \pi r^2$



when $A = 36\pi$ in^2

$\pi r^2 = 36\pi$ in^2 $\Rightarrow$ $r = 6$ in

and $\frac{dA}{dt} = \frac{1}{2} \frac{in^2}{min}$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{dA}{dt} \cdot \frac{1}{2\pi r}$

when $r = 6$ in

$$\frac{dr}{dt} = \frac{1}{2} \frac{in^2}{min} \cdot \frac{1}{12\pi \, in} = \frac{1}{24\pi} \frac{in}{min}$$

(6) Find the equation of the line tangent to the graph of $f(x) = 4x^2 - 6x$ that is parallel to the line $y = 2x$.

Slope $m = 2$. $f'(x) = 8x - 6$

$8x - 6 = 2 \Rightarrow 8x = 8 \Rightarrow x = 1$.

$f(1) = 4 - 6 = -2$. The point is $(1, -2)$.

The line is

$y + 2 = 2(x - 1)$

$y = 2x - 4$

(7) Evaluate each integral.

(a) $\displaystyle\int \frac{x + x^{7/2}}{x^3}\, dx$ $= \int (x^{-2} + x^{1/2})\, dx$

$= \dfrac{x^{-1}}{-1} + \dfrac{x^{3/2}}{3/2} + C$

$= \dfrac{-1}{x} + \frac{2}{3} x^{3/2} + C$

(b) $\displaystyle\int \frac{4x}{(x^2+1)^2}\, dx$

$u = x^2+1$

$du = 2x\,dx$

$= \int \dfrac{2\,du}{u^2} = 2 \dfrac{u^{-1}}{-1} + C$

$= -\dfrac{2}{u} + C$

$= \dfrac{-2}{x^2+1}$

(8) Evaluate each integral.

(a) $\displaystyle\int_0^{\pi/4} 6\sin(2x)\,dx$ $= -3\cos(2x)\Big|_0^{\pi/4}$

$= -3\cos\left(\frac{\pi}{2}\right) - \left(-3\cos(0)\right)$

$= 3$

(b) $\displaystyle\int \cos(\tan\theta)\sec^2\theta\,d\theta$

$u = \tan\theta$

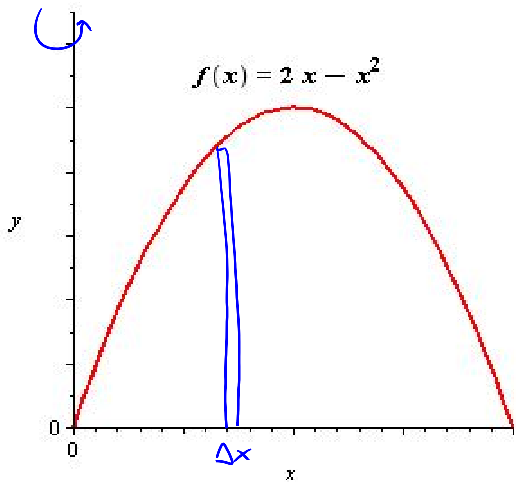
$du = \sec^2\theta\,d\theta$

$= \int \cos u\,du$

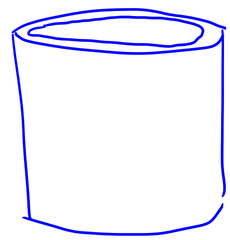
$= \sin u + C$

$= \sin(\tan\theta) + C$

(9) The first quadrant region bounded between the x-axis and the curve $f(x) = 2x - x^2$ is revolved about the y-**axis** to form a solid. Find the volume of this solid.



Shell



height

$h = f(x) = 2x - x^2$

radius

$r = x$

$$V_{shell} = 2\pi r h \, \Delta x$$

$$= 2\pi x (2x - x^2) \Delta x$$

$f(x) = 0 \Rightarrow x = 0$ or $x = 2$

$$V = 2\pi \int_0^2 (2x^2 - x^3)\, dx$$

$$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right|_0^2 = 2\pi \left[\frac{16}{3} - \frac{16}{4} \right]$$

$$= 32\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{32\pi}{12} = \frac{8\pi}{3}$$

(10) A spring has a natural length of 10 cm. It requires a force of 50 N to stretch it to a length of 20 cm ($\frac{1}{10}$ m from equilibrium).

(a) Determine the spring constant k.

$$50\,N = k\left(\frac{1}{10}\right)m \Rightarrow$$

$$k = 500\ \frac{N}{m}$$

(b) Find the work done stretching the spring from its natural 10 cm length to a length of 30 cm ($\frac{1}{5}$ m from equilibrium).

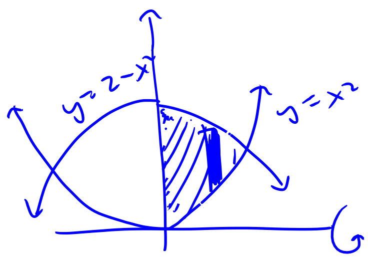
$$W = \int_0^{1/5} 500x\,dx \quad J$$

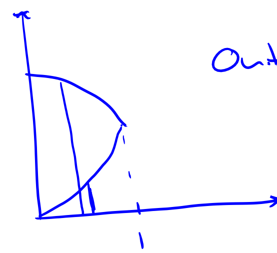
$$= 250\,x^2\Big|_0^{1/5} \quad J$$

$$= 250\left(\frac{1}{5}\right)^2 \quad J$$

$$= \frac{250}{25}\ J = 10\ J$$

(11) The first quadrant region bounded between the curves $y = 2 - x^2$ and $y = x^2$ is rotated about the x-axis to form a solid. Find the volume of this solid.





Outer radius

$R = 2 - x^2$

inner radius

$r = x^2$

$$2 - x^2 = x^2$$
$$2 = 2x^2$$
$$1 = x^2 \Rightarrow x = \pm 1$$

$$V_{washer} = (\pi R^2 - \pi r^2)\,\Delta x$$
$$= \pi\left((2-x^2)^2 - x^4\right)\Delta x$$

$$(2-x^2)^2 - x^4 = 4 - 4x^2 + x^4 - x^4 = 4 - 4x^2$$

$$V = \pi \int_0^1 (4 - 4x^2)\,dx$$
$$= 4\pi\left[x - \frac{x^3}{3}\right)\Big|_0^1$$
$$= 4\pi\left[1 - \frac{1}{3}\right] = 4\pi\left(\frac{2}{3}\right) = \frac{8\pi}{3}$$