

Final Exam Math 2253 sec. 4

Summer 2014

Name: (1 point)

Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: _____

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

INSTRUCTIONS: There are 10 problems worth 11 points each. You may eliminate any one problem, or I will count your best 9 out of 10. There are no notes, or books allowed and **no calculator is allowed**. To receive full credit, you must clearly justify your answer and use proper notation.

(1) Evaluate the derivative of each function. Do not leave compound fractions in your answer. Otherwise, it is not necessary to simplify.

$$\begin{aligned} \text{(a)} \quad f(x) &= \frac{2x-3}{x^4+x} & f'(x) &= \frac{2(x^4+x) - (2x-3)(4x^3+1)}{(x^4+x)^2} \\ & & &= \frac{-6x^4 + 12x^3 + 3}{(x^4+x)^2} \end{aligned}$$

$$\text{(b)} \quad y = x \cos(2x)$$

$$y' = \cos(2x) - 2x \sin(2x)$$

$$\text{(c)} \quad g(x) = x^3(\sqrt{x}+1) = x^{7/2} + x^3$$

$$\begin{aligned} g'(x) &= \frac{7}{2} x^{5/2} + 3x^2 \\ &= \frac{7}{2} x^2 \sqrt{x} + 3x^2 \end{aligned}$$

(2) Evaluate each definite integral.

$$\begin{aligned} \text{(a)} \quad \int_0^4 x(3x+4) dx &= \int_0^4 (3x^2 + 4x) dx \\ &= x^3 + 2x^2 \Big|_0^4 = 4^3 + 2 \cdot 4^2 - (0^3 + 2 \cdot 0^2) \\ &= 64 + 32 \\ &= 96 \end{aligned}$$

$$\text{(b)} \quad \int_0^{\frac{\pi}{2}} \sin x \cos x dx$$

$$u = \sin x \quad du = \cos x dx$$

$$\text{If } x = 0 \quad u = \sin(0) = 0$$

$$x = \frac{\pi}{2} \quad u = \sin\left(\frac{\pi}{2}\right) = 1$$

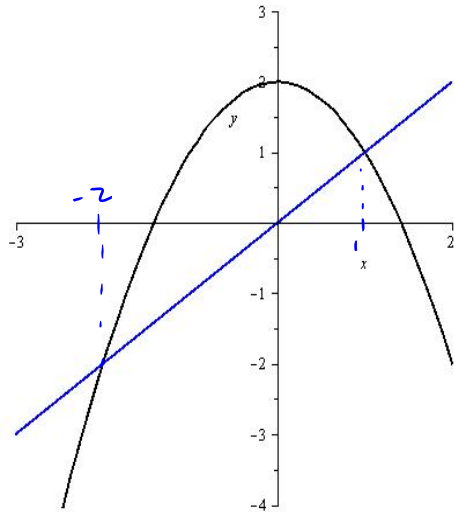
$$\begin{aligned} &= \int_0^1 u du \\ &= \frac{u^2}{2} \Big|_0^1 = \frac{1}{2} - \frac{0}{2} \\ &= \frac{1}{2} \end{aligned}$$

(3) Evaluate each indefinite integral.

$$\begin{aligned} \text{(a)} \quad \int x \sec^2(x^2) dx & \quad u = x^2 \quad du = 2x dx \\ & \quad \frac{1}{2} du = x dx \\ & = \frac{1}{2} \int \sec^2 u \, du \\ & = \frac{1}{2} \tan u + C \\ & = \frac{1}{2} \tan(x^2) + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int x^4(x^5+1)^{10} dx & \quad u = x^5+1 \quad du = 5x^4 dx \\ & \quad \frac{1}{5} du = x^4 dx \\ & = \frac{1}{5} \int u^{10} du \\ & = \frac{1}{5} \frac{u^{11}}{11} + C \\ & = \frac{1}{55} (x^5+1)^{11} + C \end{aligned}$$

(4) Find the area of the region bounded between the curves $y = 2 - x^2$ and $y = x$.



Intersection:

$$2 - x^2 = x \Rightarrow$$

$$0 = x^2 + x - 2$$

$$= (x+2)(x-1)$$

$$\Rightarrow x = -2 \text{ or } x = 1$$

The top curve is $y = 2 - x^2$

the bottom is $y = x$

$$\text{Area } A = \int_{-2}^1 (2 - x^2 - x) dx$$

$$= 2x - \frac{x^3}{3} - \frac{x^2}{2} \Big|_{-2}^1$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right)$$

$$= 2 - \frac{1}{3} - \frac{1}{2} + 4 - \frac{8}{3} + 2$$

$$= 8 - \frac{1}{2} - \frac{8}{3} = 5 - \frac{1}{2} = \frac{9}{2}$$

(5) Find all values of the constant c such that $f(x)$ is continuous at all real numbers.

$$f(x) = \begin{cases} x^2 - 2x, & x \leq 1 \\ cx - 1, & x > 1 \end{cases}$$

f is continuous for $x > 1$ and $x < 1$ since the pieces are polynomials.

$$f(1) = 1^2 - 2 \cdot 1 = -1 \quad \text{Continuity requires}$$
$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} cx - 1 = c - 1$$

$$c - 1 = -1 \Rightarrow c = 0$$

The function is continuous on all reals provided $c = 0$.

(6) Find the equation of the line tangent to the graph of the function at the indicated point. (Express your answer in the form $y = mx + b$.)

$$f(x) = x^5 + 3x^3, \quad \text{at } x = 1$$

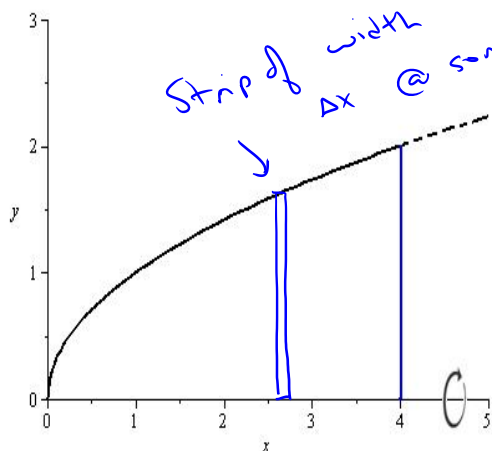
$$f(1) = 1 + 3 = 4, \quad f'(x) = 5x^4 + 9x^2$$
$$f'(1) = 5 + 9 = 14$$

The line is through $(1, 4)$ w/ slope $m = 14$.

$$y - 4 = 14(x - 1)$$

$$y = 14x - 10$$

(7) The region bounded between the x -axis, the curve $y = \sqrt{x}$, and a segment of the vertical line $x = 4$ is rotated about the x -axis. Find the volume of the resulting solid.



radius $r = y = \sqrt{x}$

The volume of one disk is

$$\begin{aligned} V_D &= \pi r^2 \Delta x \\ &= \pi (\sqrt{x})^2 \Delta x \\ &= \pi x \Delta x \end{aligned}$$

The solid has volume

$$\begin{aligned} V &= \int_0^4 \pi x \, dx = \pi \left. \frac{x^2}{2} \right|_0^4 \\ &= \pi \left(\frac{16}{2} - 0 \right) = 8\pi \end{aligned}$$

(8) Find the absolute maximum and absolute minimum values of the function f on the indicated interval.

$$f(x) = x^3 - 3x^2 - 24x + 2, \quad \text{on } [-3, 2]$$

$$f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x - 4)(x + 2)$$

The critical # of f are $x=4$ and $x=-2$.

The only one on the interval of interest is -2 .

$$\begin{aligned} f(-3) &= (-3)^3 - 3(-3)^2 - 24(-3) + 2 \\ &= -27 - 27 + 72 + 2 = 20 \end{aligned}$$

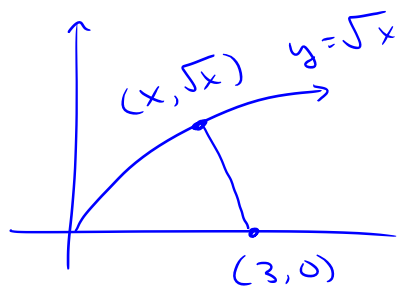
$$\begin{aligned} f(-2) &= (-2)^3 - 3(-2)^2 - 24(-2) + 2 \\ &= -8 - 12 + 48 + 2 = 30 \end{aligned}$$

$$\begin{aligned} f(2) &= 2^3 - 3(2)^2 - 24(2) + 2 \\ &= 8 - 12 - 48 + 2 = -50 \end{aligned}$$

The absolute minimum is $f(2) = -50$.

The absolute maximum is $f(-2) = 30$.

(9) Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$. (Note: minimizing the square of the distance will minimize the distance.)



For some point (x, \sqrt{x})
on the curve, the distance
to $(3, 0)$ is

$$\sqrt{(x-3)^2 + (\sqrt{x}-0)^2} = \sqrt{(x-3)^2 + x}$$

Let $f(x)$ be the squared distance

$$f(x) = (x-3)^2 + x = x^2 - 6x + 9 + x$$

$$f(x) = x^2 - 5x + 9$$

$$f'(x) = 2x - 5, \quad f'(x) = 0 \Rightarrow x = \frac{5}{2}$$

$$f''(x) = 2 \quad \text{so} \quad f''\left(\frac{5}{2}\right) = 2 > 0$$

The x -value $\frac{5}{2}$ minimizes f .

The closest point on the curve is

$$\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right).$$

(10) Let $f(t) = 6t^2 + 4t$ and define $g(x)$ by

$$g(x) = \int_1^x f(t) dt.$$

Evaluate each expression.

$$\begin{aligned} \text{(a) } g(2) &= \int_1^2 (6t^2 + 4t) dt = 2t^3 + 2t^2 \Big|_1^2 = 2 \cdot 2^3 + 2 \cdot 2^2 - (2 \cdot 1^3 + 2 \cdot 1^2) \\ &= 16 + 8 - 4 = 20 \end{aligned}$$

$$\begin{aligned} \text{(b) } g(0) &= \int_1^0 (6t^2 + 4t) dt = 2t^3 + 2t^2 \Big|_1^0 = 0 - (2 \cdot 1^3 + 2 \cdot 1^2) \\ &= -4 \end{aligned}$$

$$\text{(c) } g'(x) = 6x^2 + 4x$$

$$\text{(d) } g''(x) = 12x + 4$$