Final Exam Math 2253 sec. 4

Summer 2014

Name: (1 point) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

INSTRUCTIONS: There are 10 problems worth 11 points each. You may eliminate any one problem, or I will count your best 9 out of 10. There are no notes, or books allowed and **no calculator is allowed.** To receive full credit, you must clearly justify your answer and use proper notation.

(1) Evaluate the derivative of each function. Do not leave compound fractions in your answer. Otherwise, it is not necessary to simplify.

(a) $f(x) = \dfrac{2x-3}{x^4+x}$

$$f'(x) = \frac{2(x^4+x) - (2x-3)(4x^3+1)}{(x^4+x)^2}$$

(b) $y = x\cos(2x)$

$$y' = \cos(2x) - 2x\sin(2x)$$

(c) $g(x) = x^3(\sqrt{x}+1) = x^{7/2} + x^3$

$$g'(x) = \frac{7}{2}x^{5/2} + 3x^2$$

(2) Evaluate each definite integral.

(a) $\displaystyle\int_0^4 x(3x+4)\,dx$ $= \displaystyle\int_0^4 (3x^2 + 4x)\,dx$

$= x^3 + 2x^2 \Big]_0^4 = 4^3 + 2\cdot 4^2 - 0 = 64 - 32 = 32$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx$

$u = \sin x \quad , \quad du = \cos x\,dx$

$u(0) = 0 \qquad u(\pi/2) = 1$

$= \displaystyle\int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1$

$= \frac{1}{2} - 0 = \frac{1}{2}$

(3) Evaluate each indefinite integral.

(a) $\int x \sec^2(x^2)\,dx$

$u = x^2$ $\quad du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$= \frac{1}{2} \int \sec^2(u)\,du$

$= \frac{1}{2} \tan u + C = \frac{1}{2} \tan(x^2) + C$

(b) $\int x^4(x^5+1)^{10}\,dx$

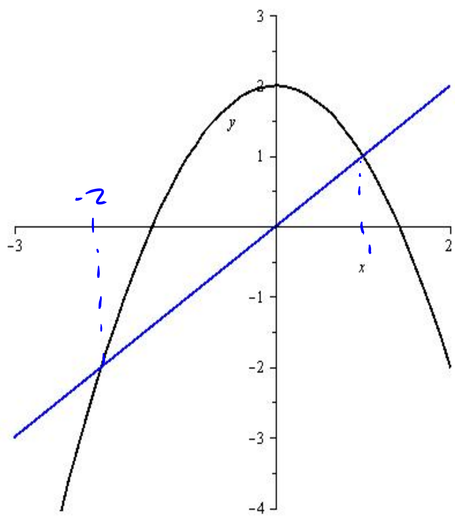
$u = x^5 + 1$, $du = 5x^4\,dx$

$\frac{1}{5} du = x^4\,dx$

$= \frac{1}{5} \int u^{10}\,du = \frac{1}{5} \cdot \frac{u^{11}}{11} + C$

$= \frac{1}{55} (x^5 + 1)^{11} + C$

(4) Find the area of the region bounded between the curves $y = 2 - x^2$ and $y = x$.



$2 - x^2 = x \Rightarrow$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = 1$ or $x = -2$

for $-2 \le x \le 1$

The parabola is on top.

$$A = \int_{-2}^{1} \left[(2 - x^2) - x \right] dx$$

$$= 2x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_{-2}^{1}$$

$$= 2(1) - \frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 - \left[2(-2) - \frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 \right]$$

$$= 2 - \frac{1}{3} - \frac{1}{2} + 4 - \frac{8}{3} + 2$$

$$= 8 - \frac{1}{2} - \frac{9}{3} = 5 - \frac{1}{2} = \frac{9}{2}$$

(5) Find all values of the constant c such that $f(x)$ is continuous at all real numbers.

$$f(x) = \begin{cases} x^2 - 2x, & x \le 1 \\ cx - 1, & x > 1 \end{cases}$$

f is continuous on $(-\infty, 1)$ and $(1, \infty)$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 2x) = 1 - 2 = -1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (cx - 1) = c - 1$ $\qquad f(1) = -1$

Continuity requires $\lim_{x \to 1} f(x) = f(1)$

Hence we require $c - 1 = -1 \Rightarrow c = 0$

f is continuous on $(-\infty, \infty)$ provided $c = 0$.

(6) Find the equation of the line tangent to the graph of the function at the indicated point. (Express your answer in the form $y = mx + b$.)

$$f(x) = x^5 + 3x^3, \quad \text{at} \quad x = 1$$

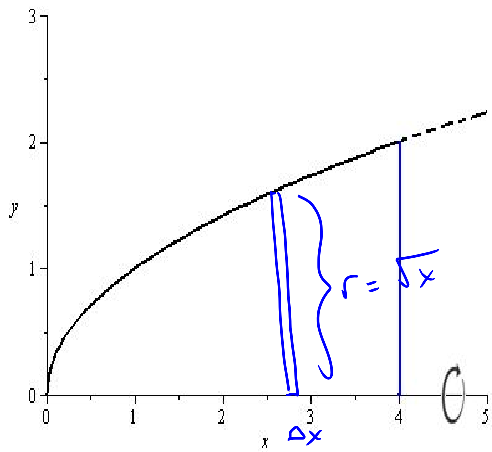
$f(1) = 1^5 + 3 \cdot 1^3 = 4$, $\quad f'(x) = 5x^4 + 9x^2 \Rightarrow f'(1) = 5 + 9 = 14$

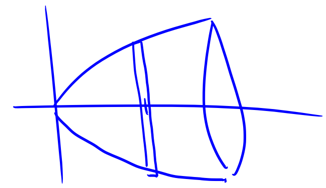
The slope $m = 14$, and the point is $(1, 4)$.

$y - 4 = 14(x - 1) \Rightarrow y = 14x - 14 + 4$

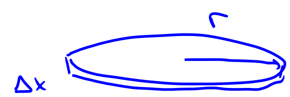
$\boxed{y = 14x - 10}$

(7) The region bounded between the x-axis, the curve $y = \sqrt{x}$, and a segment of the vertical line $x = 4$ is rotated about the x-axis. Find the volume of the resulting solid.





Take a cross section @ x between 0 and 4



$V_{disk} = \pi r^2 \Delta x$

here $r = \sqrt{x}$

The volume of one disk is $V_{disk} = \pi(\sqrt{x})^2 \Delta x = \pi x \Delta x$

Summing the disks and letting $\Delta x \to 0$, the total volume is

$$V = \int_0^4 \pi x \, dx = \frac{\pi}{2} x^2 \Big|_0^4 = \frac{\pi}{2}(16 - 0) = 8\pi$$

(8) Find the absolute maximum and absolute minimum values of the function f on the indicated interval.

$$f(x) = x^3 - 3x^2 - 24x + 2, \quad \text{on} \quad [-3, 2]$$

Critical numbers: $f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8)$

$= 3(x-4)(x+2)$

$f'(x)$ is always defined

$f'(x) = 0 \Rightarrow x = 4$ or $x = -2$.

Only -2 is inside the interval $[-3, 2]$.

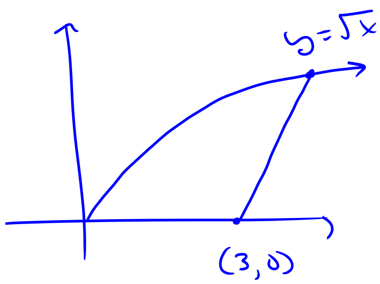
$f(-3) = (-3)^3 - 3(-3)^2 - 24(-3) + 2 = -27 - 27 + 72 + 2 = 20$

$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 2 = -8 - 12 + 48 + 2 = 30$

$f(2) = 2^3 - 3(2)^2 - 24(2) + 2 = 8 - 12 - 48 + 2 = -50$

The absolute maximum is $30 = f(-2)$. The absolute minimum is $-50 = f(2)$.

(9) Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$. (Note: minimizing the square of the distance will minimize the distance.)



The distance between $(x, \sqrt{x})$ and $(3,0)$ is

$$D = \sqrt{(x-3)^2 + (\sqrt{x}-0)^2}$$

Let $f(x) = D^2 = (x-3)^2 + x$, minimize f

$$f'(x) = 2(x-3)+1 = 2x-6+1 = 2x-5$$

$$f'(x) = 0 \Rightarrow x = \frac{5}{2}$$

$$f''(x) = 2 \Rightarrow f''(5/2) = 2 > 0$$ concave up

$\frac{5}{2}$ minimizes f by the 2^{nd} derivative test.

The closest point is therefore $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$.

(10) Let $f(t) = 6t^2 + 4t$ and define $g(x)$ by

$$g(x) = \int_1^x f(t)\,dt.$$

Evaluate each expression.

(a) $g(2)$ $= \int_1^2 (6t^2+4t)\,dt = 2t^3+2t^2 \Big|_1^2 = 2\cdot 8 + 2\cdot 4 - (2\cdot 1 + 2\cdot 1)$

$= 16 + 8 - 4 = 20$

(b) $g(0)$ $= \int_1^0 (6t^2+4t)\,dt = 2t^3+2t^2 \Big|_1^0 = 0 - (2\cdot 1+2\cdot 1) = -4$

(c) $g'(x)$ $= 6x^2 + 4x$

(d) $g''(x)$ $= 12x + 4$