Final Exam Math 2253 sec. 1

Fall 2014

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

INSTRUCTIONS: There are 11 problems worth 10 points each. You may eliminate any one problem, or I will drop the problem with your lowest score. There are no notes, or books allowed and **no calculator is allowed. Illicit use of a calculator, smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam and may result in a formal reported allegation of academic misconduct.** To receive full credit, you must clearly justify your answer and use proper notation.

(1) Evaluate each limit. (Use of l'Hopital's rule will not be considered for credit.)

(a) $\lim_{x\to 0} \frac{\tan(3x)}{x}$ $= \lim_{x\to 0} 3 \frac{\tan 3x}{3x}$

$= \lim_{x\to 0} \frac{3}{\cos 3x} \frac{\sin 3x}{3x} = \frac{3}{1}(1) = 3$

(b) $\lim_{t\to\infty} \frac{\sqrt{4t^2+1}}{t-2}$ $= \lim_{t\to\infty} \frac{\sqrt{4t^2+1}}{t-2} \cdot \frac{\frac{1}{t}}{\frac{1}{t}}$

$\frac{1}{t} = \frac{1}{\sqrt{t^2}}$ for $t>0$

$= \lim_{t\to\infty} \frac{\sqrt{4+1/t^2}}{1-2/t} = \frac{\sqrt{4+0}}{1-0} = 2$

(c) $\lim_{x\to -3} \frac{x^2-9}{x+3}$ $= \lim_{x\to -3} \frac{(x-3)(x+3)}{x+3}$

$= \lim_{x\to -3} x-3 = -3-3 = -6$

(2) Consider the function $f(x) = \begin{cases} \frac{\sqrt{x^2+1}-1}{x^2}, & x \neq 0 \\ c, & x = 0. \end{cases}$

(a) Evaluate $\lim_{x\to 0} f(x)$ $= \lim_{x\to 0} \frac{\sqrt{x^2+1} - 1}{x^2} \cdot \frac{\sqrt{x^2+1} + 1}{\sqrt{x^2+1} + 1}$

$$= \lim_{x\to 0} \frac{(x^2+1) - 1}{x^2(\sqrt{x^2+1} + 1)}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+1} + 1)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+1} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

(b) Determine the value of c such that f is continuous at zero.

f is cont. @ 0 if

$$\lim_{x\to 0} f(x) = f(0)$$

$$\frac{1}{2} = c$$

(3) Find $\frac{dy}{dx}$. Do not leave compound fractions in your answers.

(a) $y = x^3(2x-1)^4$

$$\frac{dy}{dx} = 3x^2(2x-1)^4 + x^3\left(4(2x-1)^3 \cdot 2\right)$$

$$= 3x^2(2x-1)^4 + 8x^3(2x-1)^3$$

$$= x^2(2x-1)^3\left(3(2x-1) + 8x\right)$$

$$= x^2(2x-1)^3(14x-3)$$

(b) $y = \dfrac{x^2+4}{x+3}$

$$\frac{dy}{dx} = \frac{2x(x+3) - (x^2+4)}{(x+3)^2} = \frac{x^2+6x-4}{(x+3)^2}$$

(c) $x^2 + 2xy - y^3 = 1$

$$2x + 2y + 2xy' - 3y^2y' = 0$$

$$(2x - 3y^2)\frac{dy}{dx} = -(2x+2y)$$

$$\frac{dy}{dx} = \frac{-(2x+2y)}{2x-3y^2}$$

(4) State the derivative of each of the six trigonometric functions.

$\frac{d}{dx} \sin x = $ Cos x

$\frac{d}{dx} \cos x = $ - Sin x

$\frac{d}{dx} \sec x = $ Sec x tan x

$\frac{d}{dx} \csc x = $ - Csc x Cot x

$\frac{d}{dx} \tan x = $ $Sec^2 x$

$\frac{d}{dx} \cot x = $ $-Csc^2 x$

(5) Find the equation of the line tangent to the graph of $f(x) = (6x - 5)^4$ at the point where $x = 1$.

$f(1) = (6-5)^4 = 1$ $\quad$ $f'(x) = 4(6x-5)^3 \cdot 6$

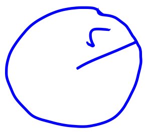
Slope $\quad m = f'(1) = 24(6-5)^3 = 24$

$y - 1 = 24(x-1)$

$\boxed{y = 24x - 23}$

(6) Oil drips from a car leaving a circular puddle. At the moment that the area of the circular puddle is 36π square inches, its area is increasing at a rate of $1/2$ square inches per minute. Determine the rate at which the radius is increasing at this moment. (Your answer should include appropriate units.)

For radius r and area A



$$A = \pi r^2$$

given $\frac{dA}{dt} = \frac{1}{2} \frac{in^2}{min}$ find

$\frac{dr}{dt}$ when $A = 36\pi \ in^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt}$$

when $A = 36\pi \ in^2$

$$\pi r^2 = 36\pi \ in^2 \implies r^2 = 36 \ in^2$$

$$\implies r = 6 \ in$$

At this moment

$$\frac{dr}{dt} = \frac{1}{2\pi \, 6 \, in} \cdot \frac{1}{2} \frac{in^2}{min}$$

$$= \frac{1}{24\pi} \frac{in}{min}$$

(7) Evaluate each integral.

(a) $\displaystyle\int \frac{x + x^{7/2}}{x^3}\, dx$ $= \int (x^{-2} + x^{1/2})\, dx$

$$= \frac{x^{-1}}{-1} + \frac{x^{3/2}}{3/2} + C$$

$$= \frac{-1}{x} + \frac{2}{3} x^{3/2} + C$$

(b) $\displaystyle\int \frac{4t}{\sqrt{t^2+1}}\, dt$

$u = t^2 + 1$ $\quad du = 2t\,dt$

$2du = 4t\,dt$

$$= \int \frac{2du}{\sqrt{u}} = 2\int u^{-1/2}\, du$$

$$= 2\,\frac{u^{1/2}}{1/2} + C$$

$$= 4\sqrt{u} + C$$

$$= 4\sqrt{t^2+1} + C$$

(8) Evaluate each integral.

(a) $\int_0^{\pi/6} 9\cos(3x)\,dx$

$u = 3x \qquad du = 3dx$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{6} \Rightarrow u = \frac{\pi}{2}$

$= 3\int_0^{\pi/2} \cos u\, du$

$= 3\sin u \Big|_0^{\pi/2} = 3\sin\frac{\pi}{2} - 3\sin 0$

$= 3 - 0$

$= 3$

(b) $\int \sin(\cot\theta)\csc^2\theta\,d\theta$

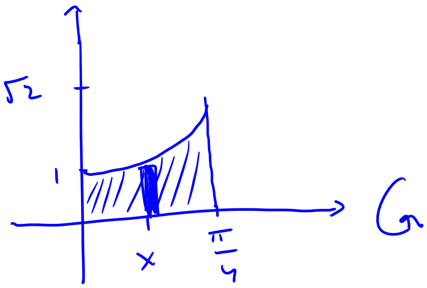
$u = \cot\theta \qquad du = -\csc^2\theta\,d\theta$

$= -\int \sin(u)\,du$

$= \cos u + C$

$= \cos(\cot\theta) + C$

(9) The region bounded between the curve $y = \sec x$ and the x-axis on the interval $0 \leq x \leq \frac{\pi}{4}$ is rotated about the x-axis to form a solid. Find the volume of this solid.



The radius of one disk @ x is

$r = y = \sec x$

Volume of disk

$V_D = \pi r^2 \Delta x = \pi (\sec x)^2 \Delta x$

The whole volume is

$$V = \int_0^{\pi/4} \pi \sec^2 x \, dx$$

$$= \pi \tan x \Big|_0^{\pi/4}$$

$$= \pi \tan \pi/4 - \pi \tan 0 = \pi$$

(10) A spring has a natural length of 20 cm. It requires a force of 40 N to stretch it to a length of 30 cm ($\frac{1}{10}$ m from equilibrium).

(a) Determine the spring constant k.

$$F = kx$$

$$40\,N = k\left(\tfrac{1}{10}\,m\right) \implies k = 400\ \frac{N}{m}$$

(b) Find the work done stretching the spring from its natural 20 cm length to a length of 40 cm ($\frac{1}{5}$ m from equilibrium).

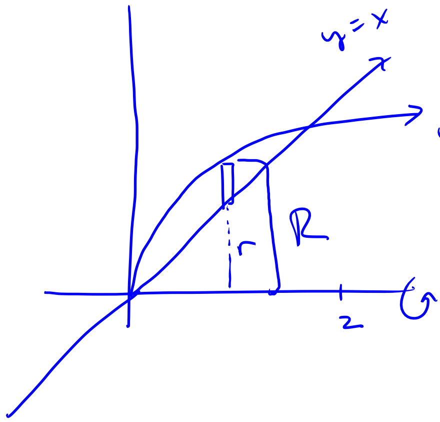
$$W = \int_0^{1/5} 400\,x\,dx \quad Nm$$

$$= 200x^2 \Big|_0^{1/5} J = 200\left(\tfrac{1}{5}\right)^2 - 200\,(0)^2 \quad J$$

$$= 200 \cdot \frac{1}{25} \quad J$$

$$= 8\ J$$

(11) The region bounded between the curves $y = x$ and $y = \sqrt{2x}$ is rotated about the x-axis to form a solid. Find the volume of this solid.



Intersection:

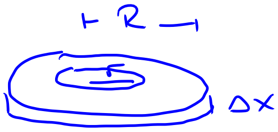
$$x = \sqrt{2x}$$

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0 \Rightarrow \quad x = 0 \text{ or } x = 2$$

Outer radius $R = \sqrt{2x}$, inner radius $r = x$



Volume of one washer (disk - disk)

$$V_w = \pi R^2 \Delta x - \pi r^2 \Delta x$$

$$= \pi \left((\sqrt{2x})^2 - x^2 \right) \Delta x$$

$$= \pi \left(2x - x^2 \right) \Delta x$$

The volume is

$$V = \int_0^2 \pi \left(2x - x^2\right) dx$$

$$= \pi \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = \pi \left(4 - \frac{8}{3} \right)$$

$$= \pi \left(\frac{12-8}{3} \right) = \frac{4\pi}{3}$$