Final Exam Math 2254 sec. 2

Fall 2014

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

INSTRUCTIONS: There are 11 problems worth 10 points each. You may eliminate any one problem, or I will drop the problem with your lowest score. There are no notes, or books allowed and **no calculator is allowed. Illicit use of a calculator, smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam and may result in a formal reported allegation of academic misconduct.** To receive full credit, you must clearly justify your answer and use proper notation.

(1) Evaluate the derivative of each function.

(a) $f(x) = xe^{x^3}$

$$f'(x) = e^{x^3} + 3x^3 e^{x^3}$$

(b) $g(t) = \log_4(t - \cos(t))$

$$g'(t) = \frac{1}{\ln 4} \quad \frac{1 + \sin t}{t - \cos t}$$

(c) $f(x) = \sin^{-1}(e^x)$

$$f'(x) = \frac{e^x}{\sqrt{1 - e^{2x}}}$$

(2) Evaluate each indefinite integral.

(a) $\displaystyle\int \frac{8}{x^2+16}\,dx$ $= \frac{8}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$

$= 2\tan^{-1}\left(\frac{x}{4}\right)+C$

(b) $\displaystyle\int \frac{2x}{x^2+16}\,dx$ $= \displaystyle\int \frac{du}{u}$

$u = x^2+16$

$du = 2x\,dx$

$= \ln|u| + C$

$= \ln|x^2+16| + C$

(3) Evaluate the integral using any applicable method.

$$\int x \ln(x)\, dx$$

$u = \ln x \qquad du = \frac{dx}{x}$

$v = \frac{x^2}{2} \qquad dv = x\, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

(4) Evaluate the integral using any applicable method.

$$\int \frac{4x^2}{\sqrt{1-x^2}}\,dx$$

$$x = \sin\theta$$

$$dx = \cos\theta \, d\theta$$

$$\sqrt{1-x^2} = \cos\theta$$

$$= 4 \int \frac{\sin^2\theta \cos\theta \, d\theta}{\cos\theta}$$

$$= 4 \int \sin^2\theta \, d\theta$$

$$= 2 \int (1 - \cos 2\theta)\, d\theta$$

$$= 2\theta - \sin 2\theta + C$$

$$= 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}x - 2x\sqrt{1-x^2} + C$$

(5) Evaluate the integral using any applicable method.

$$\int \frac{x-2}{x^2(x-1)}\,dx$$

$$\frac{x-2}{x^2(x-1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1}$$

$$x-2 = A(x-1) + Bx(x-1) + Cx^2$$

$x=0$

$$-2 = -A \Rightarrow A = 2$$

$x=1 \quad -1 = C$

$x=2 \quad 0 = A + 2B + 4C$

$$0 = 2 + 2B - 4 \Rightarrow 2B - 2 = 0$$

$$B = 1$$

$$\int \frac{x-2}{x^2(x-1)}\,dx = \int\left(\frac{2}{x^2} + \frac{1}{x} - \frac{1}{x-1}\right)dx$$

$$= \frac{-2}{x} + \ln|x| - \ln|x-1| + C$$

(6) (a) Determine whether the integral is proper or improper. If improper, briefly state why.

$$\int_0^2 \frac{dx}{(x-1)^3}$$

it's improper

the integrand is undefined @ 1

(b) Evaluate the integral if possible or show that it is divergent.

$$\int_0^2 \frac{dx}{(x-1)^3} = \int_0^1 \frac{dx}{(x-1)^3} + \int_1^2 \frac{dx}{(x-1)^3} \quad \text{if defined}$$

$$\int_0^1 \frac{dx}{(x-1)^3} = \lim_{t \to 1^-} \int_0^t \frac{dx}{(x-1)^3}$$

$$= \lim_{t \to 1^-} \left. \frac{-1}{2(x-1)^2} \right|_0^t$$

$$= \lim_{t \to 1^-} \left(\frac{-1}{2(t-1)^2} - \frac{-1}{2} \right) = -\infty$$

$$\int_0^2 \frac{dx}{(x-1)^3} \quad \text{is divergent.}$$

(7) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Use any applicable method.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$

$0 \le \tan^{-1}(n) \le \pi/2 \quad \forall n \geq 1$

all nonnegative terms

$0 \le \dfrac{\tan^{-1}(n)}{n^{3/2}} \le \dfrac{\pi/2}{n^{3/2}}$

As $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series,

this series converges absolutely by

direct comparison.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{5}{2}\right)^n$

Geometric w/ $r = \frac{-5}{2}$

$|r| = \frac{5}{2} > 1$

It is divergent.

(8) Find a Taylor series for the function $f(x) = e^{3x}$ **centered at** $a = 1$ using any applicable method.

$$e^{3x} = e^{3(x-1+1)} = e^{3} e^{3(x-1)}$$

$$= e^{3} \sum_{n=0}^{\infty} \frac{[3(x-1)]^{n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^{3} 3^{n} (x-1)^{n}}{n!}$$

(9) Find the equation of the line tangent to the parametric curve at the indicated point.

(a) $x = t^2 - 3, \quad y = e^{2t}, \quad$ at $\quad (-2, e^2)$

$-2 = t^2 - 3 \Rightarrow t^2 = 1 \quad t = 1$ or $t = -1$

$e^{2t} = e^2 \Rightarrow t = 1$

so $t = 1$

$x'(t) = 2t \qquad x'(1) = 2$

$y'(t) = 2e^{2t} \qquad y'(1) = 2e^2$

$m = \frac{2e^2}{2} = e^2$

$y - e^2 = e^2(x+2)$

$y = e^2 x + 3e^2$

(b) $x = \theta \cos\theta, \quad y = \theta \sin\theta, \quad$ at $\quad \theta = 2\pi$

$x' = \cos\theta - \theta \sin\theta \qquad y' = \sin\theta + \theta\cos\theta$

$x'(2\pi) = 1 \qquad y'(2\pi) = 2\pi$

$m = \frac{2\pi}{1} = 2\pi$

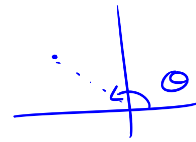
$x(2\pi) = 2\pi \qquad y(2\pi) = 0$

$y = 2\pi(x - 2\pi)$

$y = 2\pi x - 4\pi^2$

(10) (a) Convert the Cartesian point $(x, y) = (-1, 1)$ to polar coordinates (r, θ).

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ $\qquad$ $\tan\theta = \frac{1}{-1} = -1$



$\theta = \frac{3\pi}{4}$

$\left(\sqrt{2}, \frac{3\pi}{4}\right)$

(b) Find the distance between the points in the plane with polar coordinates $\left(\sqrt{2}, \frac{\pi}{4}\right)$, and $\left(3, \frac{3\pi}{2}\right)$.

(Hint: The standard distance formula only applies when points are known in Cartesian coordinates.)

$x = \sqrt{2}\cos\frac{\pi}{4} = 1$, $y = \sqrt{2}\sin\frac{\pi}{4} = 1$ $\qquad (1, 1)$

$x = 3\cos\frac{3\pi}{2} = 0$, $y = 3\cos\frac{3\pi}{2} = -3$ $\qquad (0, -3)$

$d = \sqrt{(1-0)^2 + (1+3)^2} = \sqrt{1 + 16} = \sqrt{17}$

(11) Find the area bounded by the polar curve[1] $r = 2(1 + \cos\theta)$ for $0 \le \theta \le \frac{\pi}{2}$.

$$A = \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 4 (1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

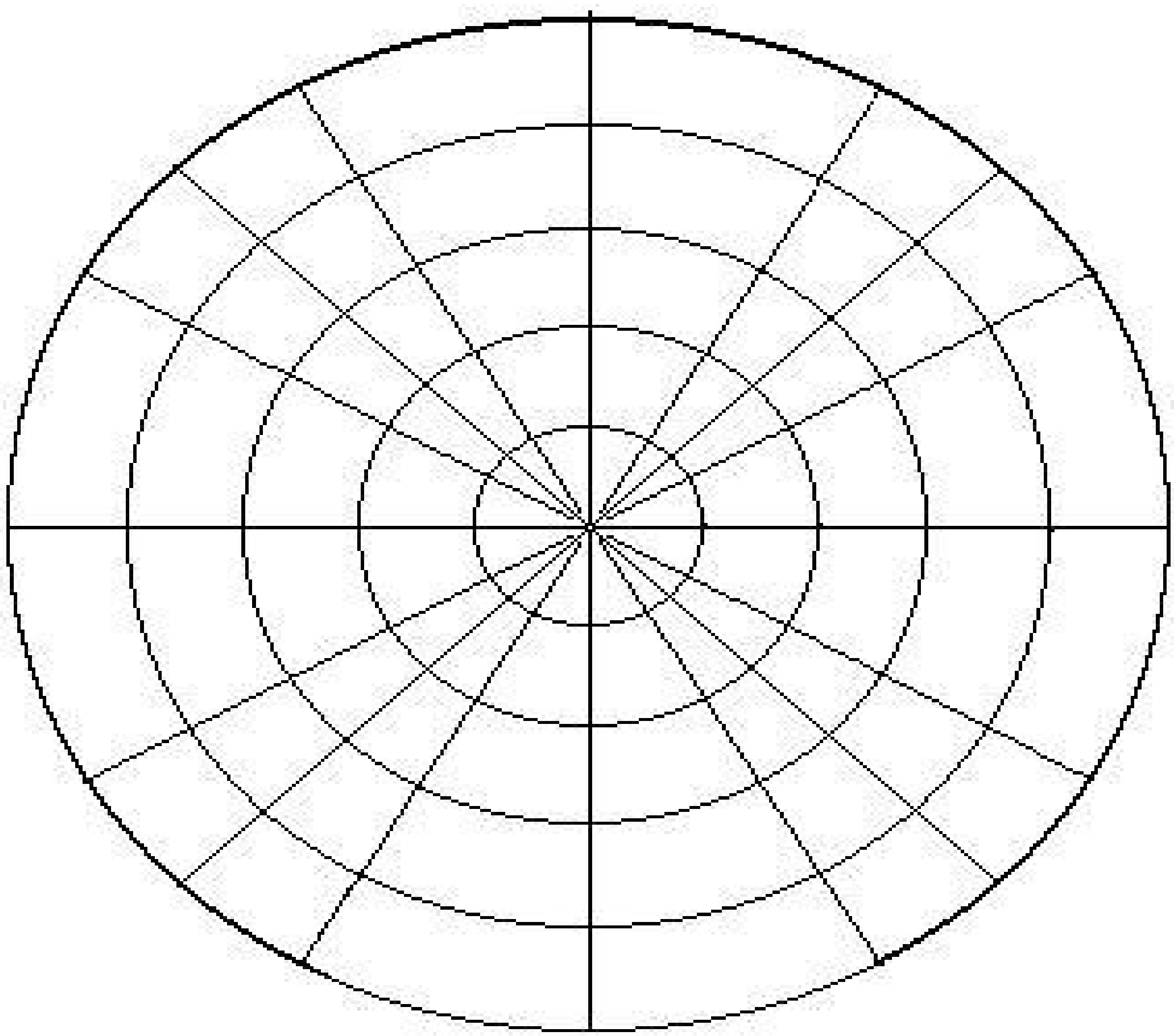
$$= 2 \int_0^{\pi/2} (1 + 2\cos\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta) \, d\theta$$

$$= 2 \left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta \right|_0^{\pi/2}$$

$$= 2 \left[\frac{3\pi}{4} + 2 + 0 \right] = \frac{3\pi}{2} + 4$$

[1] If you wish to plot the curve, polar graph paper is provided on the next page.

Polar Graph Paper



Some Potentially Useful Results

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all x}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all x}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for all x}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x \le 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 < x \le 1$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad |x| < 1$$

$$\text{where} \quad \binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$