Final Exam Math 2335 sec. 1

Fall 2009

Name: (4pts) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 16 points each; you can eliminate any one problem, or I will count your best 6 out of 7 ($6 \times 16 + 4 = 100$). You may use your text book (Atkinson & Han), one 8.5" by 11" page of written notes and a calculator. To receive full credit, you must clearly justify your answers.

(1) Find the condition number for the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \qquad A^{-1} = \frac{1}{16-1}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{4}{15} & \frac{-1}{15} \\ \frac{-1}{15} & \frac{4}{15} \end{bmatrix}$$

$$\text{Cond}(A) = \|A\| \, \|A^{-1}\|$$

$$\|A\| = \max\{5,5\} = 5 \qquad \|A^{-1}\| = \max\left\{\frac{5}{15}, \frac{5}{15}\right\} = \frac{1}{3}$$

$$\text{Cond}(A) = 5 \cdot \frac{1}{3} = \frac{5}{3}$$

(2) Let α be the unique root of the function $f(x) = 2 - x - \ln x$. Find an interval $[a, b]$ containing α. Estimate the minimum number of iterations needed to find α using the bisection method within an accuracy of 10^{-10}.

The domain of f is $(0,\infty)$ due to the log.

$f(1) = 2 - 1 - \ln 1 = 1 > 0 \qquad f(2) = 2 - 2 - \ln 2 = -\ln 2 < 0$

So a root exists on $[1,2]$. Bisection requires

$$n \geq \frac{\ln\left(\frac{b-a}{\varepsilon}\right)}{\ln 2} = \frac{\ln\left(\frac{2-1}{10^{-10}}\right)}{\ln 2} \doteq 33.22$$

34 iterations will give the required accuracy for $[a,b] = [1,2]$.

(3) Find the unique polynomial of degree ≤ 2 that passes through the points $(0,3)$, $(1,1)$, and $(2,2)$.

$x_0 = 0 \qquad x_1 = 1 \qquad x_2 = 2$

$y_0 = 3 \qquad y_1 = 1 \qquad y_2 = 2$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{2}(x-1)(x-2) = \frac{1}{2}(x^2-3x+2)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -x(x-2) = -(x^2-2x)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{2}x(x-1) = \frac{1}{2}(x^2-x)$$

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$$= \frac{3}{2}(x^2-3x+2) - (x^2-2x) + \frac{2}{2}(x^2-x)$$

$$= \frac{3}{2}x^2 - \frac{9}{2}x + 3 - x^2 + 2x + x^2 - x$$

$$\boxed{P_2(x) = 3/2\, x^2 - \frac{7}{2}x + 3}$$

Divided Differences could be used instead.

(4) Approximate the integral (present your answers showing 10 digits to the right of the decimal)

$$\int_{-1}^{1} \frac{1}{1+x^4}\,dx$$

(a) using $T_2(f)$, the trapazoid rule with two subintervals,

(b) using $T_4(f)$, the trapazoid rule with four subintervals, and

(c) using $R_4(f)$ the Richardson extrapolation method for the trapazoid rule.

a) T_2 : $h = \frac{1-(-1)}{2} = 1$ $\quad x_0 = -1 \quad x_1 = 0 \quad x_2 = 1$

$$T_2(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + 2\cdot 1 + \frac{1}{2}\right] = \frac{3}{2} = 1.5000000000$$

b) T_4 : $h = \frac{1-(-1)}{4} = \frac{1}{2}$ $\quad x_0 = -1,\ x_1 = -\frac{1}{2},\ x_2 = 0,\ x_3 = \frac{1}{2},\ x_4 = 1$

$$T_4(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{4}\left[\frac{1}{2} + 2\cdot\frac{16}{17} + 2\cdot 1 + 2\cdot\frac{16}{17} + \frac{1}{2}\right]$$

$$\doteq 1.6911764706$$

c) $R_4 = \frac{1}{3}\left[4T_4 - T_2\right]$

$$\doteq \frac{1}{3}\left[5.2647058824\right]$$

$$\doteq 1.7549019608$$

(5) Find the fourth order Taylor polynomial with the remainder term for

$$f(x) = \sqrt{1-x} \quad \text{centered at } a = 0.$$

$f(x) = \sqrt{1-x}$ $\qquad$ $f(0) = 1$

$f'(x) = \frac{-1}{2}(1-x)^{-1/2}$ $\qquad$ $f'(0) = \frac{-1}{2}$

$f''(x) = \frac{-1}{4}(1-x)^{-3/2}$ $\qquad$ $f''(0) = \frac{-1}{4}$

$f'''(x) = \frac{-3}{8}(1-x)^{-5/2}$ $\qquad$ $f'''(0) = \frac{-3}{8}$

$f^{(4)}(x) = \frac{-15}{16}(1-x)^{-7/2}$ $\qquad$ $f^{(4)}(0) = \frac{-15}{16}$

$f^{(5)}(x) = \frac{-105}{32}(1-x)^{-9/2}$ $\qquad$ $f^{(5)}(c) = \frac{-105}{32}(1-c)^{-9/2}$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$R_4(x) = \frac{x^5}{5!} f^{(5)}(c)$$ for some c between 0 and x

$$P_4(x) = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^2 - \tfrac{5}{128}x^4$$

$$R_4(x) = -\frac{7x^5}{256}(1-c)^{-9/2}$$

(6) Consider using a fourth order polynomial $P_4(x)$ to approximate the function

$$f(x) = \sin(\pi x)$$

on the interval $-1 \leq x \leq 1$.

(a) Find the x-values x_0, x_1, x_2, x_3, x_4 that will minimize the error.

(b) Use the fact that $|f^{(5)}(x)| \leq 307$ for $-1 \leq x \leq 1$ to bound the error $|f(x) - P_4(x)|$.

a) The Chebyshev nodes are

$$x_j = \cos\left(\frac{(2j+1)\pi}{2\cdot 5}\right) \quad j = 0, 1, \ldots, 4$$

$$x_0 = \cos\left(\frac{\pi}{10}\right) \doteq 0.951057$$

$$x_1 = \cos\left(\frac{3\pi}{10}\right) \doteq 0.587785$$

$$x_2 = \cos\left(\frac{5\pi}{10}\right) = 0$$

$$x_3 = \cos\left(\frac{7\pi}{10}\right) \doteq -0.587785$$

$$x_4 = \cos\left(\frac{9\pi}{10}\right) \doteq -0.951057$$

b) Error $\quad |f(x) - P_4(x)| \leq \frac{L}{2^4}$ where $L = \max_{[-1,1]} \left| \frac{f^{(5)}(x)}{5!} \right|$

Here, $L = \frac{307}{120}$

So $|f(x) - P_4(x)| \leq \frac{307}{120} \cdot \frac{1}{16} \doteq 0.159896$

(7) Find an LU decomposition of the given matrix.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 - \frac{1}{2}R_1 \to R_2 \qquad m_{21} = \frac{1}{2}$

$R_3 - 0R_1 \to R_3 \qquad m_{31} = 0$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 - \frac{2}{3}R_2 \to R_3 \qquad m_{32} = \frac{2}{3}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 2/3 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

Scratch

$$\begin{array}{ccc} 1 & 2 & 1 \\ -1 & -1/2 & 0 \\ \hline 0 & 3/2 & 1 \end{array}$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ 0 & -1 & -2/3 \\ \hline 0 & 0 & 4/3 \end{array}$$

Verification:

$$LU = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4/2 & 1 \\ 0 & 1 & \frac{6}{3} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = A$$