Final Exam Math 2254 sec. 9

Spring 2012

Name: (1 point) w/ Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

INSTRUCTIONS: There are 10 problems worth 11 points each. You may eliminate any one problem, or I will count your best 9 out of 10. You may use a TI-83/84 calculator or a scientific calculator without graphing capabilities. **A smart device may not substitute for a calculator.** There are no notes, or books allowed. **Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam.** To receive full credit, you must clearly justify your answer.

(1) Find the derivative of the given function.

(a) $f(x) = e^{2x} \ln(x^3 + 2x)$

$$f'(x) = 2e^{2x} \ln(x^3 + 2x) + e^{2x}\left(\frac{3x^2 + 2}{x^3 + 2x}\right)$$

(b) $g(t) = \tan^{-1} t^2$

$$g'(t) = \frac{1}{1 + (t^2)^2} \cdot 2t = \frac{2t}{1 + t^4}$$

(2) (a) Determine if the integral is proper or improper. If improper, state why.

$$\int_0^1 xe^{\frac{x}{2}}\,dx$$

It's proper.

(b) Evaluate the integral if possible.

$$\int_0^1 x e^{x/2}\,dx$$

By parts

$u = x$ $\quad du = dx$

$dv = e^{x/2}\,dx$ $\quad v = 2e^{x/2}\,dx$

$$= 2x e^{x/2}\Big|_0^1 - \int_0^1 2e^{x/2}\,dx$$

$$= 2x e^{x/2}\Big|_0^1 - 4e^{x/2}\Big|_0^1$$

$$= 2e^{\frac{1}{2}} - 2\cdot 0 e^{0} - 4\left(e^{\frac{1}{2}} - e^{0}\right)$$

$$= 2e^{1/2} - 4e^{\frac{1}{2}} + 4 = 4 - 2e^{1/2}$$

(3) (a) Determine if the integral is proper or improper. If improper, state why.

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}}$$

Improper. The integrand is undefined @ the upper limit 2.

(b) Evaluate the integral if possible.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{t\to 2^-} \int_0^t \frac{dx}{\sqrt{4-x^2}}$$

$$= \lim_{t\to 2^-} \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^t$$

$$= \lim_{t\to 2^-} \left(\sin^{-1}\left(\frac{t}{2}\right) - \sin^{-1}(0)\right)$$

$$= \sin^{-1}\left(\frac{2}{2}\right) - 0 = \frac{\pi}{2}$$

(4) Find the radius and interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n\, n^4}$$

Ratio test: $a_n = \dfrac{(x-1)^n}{3^n n^4}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{\cancel{n}+1}}{\cancel{3^{n}}3^{+1}(n+1)^4} \cdot \frac{\cancel{3^n}\, n^4}{\cancel{(x-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-1)\; n^4}{3\,(n+1)^4} \right| = \lim_{n\to\infty} \frac{|x-1|}{3} \left(\frac{n}{n+1} \right)^4$$

$$= \frac{|x-1|}{3} \cdot 1 = \frac{|x-1|}{3} \qquad L = \frac{|x-1|}{3}$$

The series converges absolutely if $\dfrac{|x-1|}{3} < 1$

$\Rightarrow \; |x-1| < 3$ The radius $R = 3$.

$-3 < x-1 < 3 \;\Rightarrow\; -2 < x < 4$

Check ends: $x = -2$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-2-1)^n}{3^n n^4} = \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

Convergent alt. Series

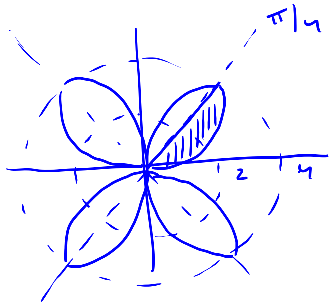
$x = 4$ $\displaystyle\sum_{n=1}^{\infty} \frac{(4-1)^n}{3^n n^4} = \sum_{n=1}^{\infty} \frac{3^n}{3^n n^4} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ p-series $p = 4 > 1$

Convergent p-series

The interval is $-2 \le x \le 4$.

a.k.a. $[-2, 4]$.

(5) Find the area bounded by the polar curve $r = 4\sin(2\theta)$ within the sector $0 \le \theta \le \frac{\pi}{4}$.



$\frac{1}{2}$ of one petal of a 4-petal rose

Area $\quad A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

$$A = \frac{1}{2}\int_0^{\pi/4} (4\sin(2\theta))^2 \, d\theta$$

$$= \frac{16}{2}\int_0^{\pi/4} \sin^2(2\theta) \, d\theta$$

$$= 8\int_0^{\pi/4} \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right) d\theta$$

$$= 4\int_0^{\pi/4} (1 - \cos(4\theta)) \, d\theta$$

$$= 4\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big|_0^{\pi/4}$$

$$= 4\left(\frac{\pi}{4} - \frac{1}{4}\sin(\pi) - \left(0 - \frac{1}{4}\sin(0)\right)\right)$$

$$= 4\left(\frac{\pi}{4} - 0\right) = \pi$$

(6) Find the Taylor polynomial of degree 2 centered at $a = 0$ of the function

$f(x) = \sqrt{1+x}$ $\qquad$ $f(0) = 1$

$$f'(x) = \frac{1}{2\sqrt{1+x}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{4}(1+x)^{-3/2} \qquad f''(0) = \frac{-1}{4}$$

$$T_2(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$T_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

(7) Find the equation of the line tangent to the parametric curve at the given value of the parameter.

$$x = t\cos t, \quad y = t\sin t; \quad t = \pi$$

$$\frac{dx}{dt} = \cos t - t\sin t \qquad \frac{dy}{dt} = \sin t + t\cos t$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t + t\cos t}{\cos t - t\sin t}$$

when $t = \pi$

$$x = \pi\cos(\pi) = -\pi \qquad y = \pi\sin(\pi) = 0$$

and slope $m = \left.\frac{dy}{dx}\right|_{t=\pi} = \frac{\sin(\pi) + \pi\cos(\pi)}{\cos(\pi) - \pi\sin(\pi)} = \frac{-\pi}{-1} = \pi$

The line is

$$y - 0 = \pi(x - (-\pi))$$

$$y = \pi x + \pi^2$$

(8) Evaluate the improper integral, or determine that it is divergent.

$$\int_0^\infty \frac{\tan^{-1} x}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{\tan^{-1} x}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\tan^{-1} x\right)^2 \Big|_0^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\tan^{-1} t\right)^2 - \frac{1}{2}\left(\tan^{-1} 0\right)^2$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$$

$$* \quad \int \frac{\tan^{-1} x}{1+x^2}\,dx \qquad u = \tan^{-1} x, \quad du = \frac{1}{1+x^2}\,dx$$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C = \frac{1}{2}\left(\tan^{-1} x\right)^2 + C$$

(9) Evaluate the indefinite integral.

$$\int \frac{3x-4}{x^2-3x+2}\,dx$$

Patical Fracton:

$$\frac{3x-4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$3x-4 = A(x-2) + B(x-1)$$

$x=1 \qquad -1 = -A \Rightarrow A = 1$

$x=2 \qquad 2 = B \Rightarrow B = 2$

$$\int \frac{3x-4}{x^2-3x+2}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{x-2}\right) dx$$

$$= \ln|x-1| + 2\ln|x-2| + C$$

(10) Evaluate the given limit. If you use any theorems (Squeeze theorem, l'Hospital's rule, etc.), identify them.

$$\lim_{x \to 0} \frac{1 - \cos x}{x^3 - x^2} = \text{"}\frac{0}{0}\text{"} \qquad \text{Use l'H rule}$$

$$= \lim_{x \to 0} \frac{\sin x}{3x^2 - 2x} = \text{"}\frac{0}{0}\text{"} \qquad \text{again}$$

$$= \lim_{x \to 0} \frac{\cos x}{6x - 2} = \frac{1}{-2} = -\frac{1}{2}$$

Potentially Useful Formulas.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad -1 < x \leq 1$$

$$\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n, \quad |r| < 1$$