

Make Up Final Exam Math 2306 sec. 006

Spring 2013

Name: _____

Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: _____

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 16 points each. **You must eliminate any one problem.** The remaining problems will be graded. If you don't indicate which problem is to be eliminated, I will count the first 6 problems (as they appear numbered herein). You may use your one sheet (8.5" x 11") of notes. No use of the text is allowed.

Illicit use of a book or any smart device will result in a grade of zero on this exam. Show all of your work on the paper provided to receive full credit.

(1) Solve the initial value problem using any applicable method. An implicit solution is acceptable.

$$\ln x \, dx - 2xy \, dy = 0 \quad y(1) = 3$$

Separate the Variables

$$\ln x \, dx = 2xy \, dy \Rightarrow 2y \, dy = \frac{\ln x}{x} \, dx$$

$$\int 2y \, dy = \int \frac{\ln x}{x} \, dx \quad \text{if } u = \ln x, \text{ then } du = \frac{1}{x} \, dx$$

$$y^2 = \frac{1}{2} (\ln x)^2 + C$$

Using the initial condition

$$3^2 = \frac{1}{2} (\ln 1)^2 + C = 0 + C \Rightarrow C = 9$$

$$y^2 = \frac{1}{2} (\ln x)^2 + 9$$

(2) A 4 pound weight stretches a spring 2 ft.

(a) Find the spring constant k .

$$F = kx \quad 4 \text{ lb} = k(2 \text{ ft}) \Rightarrow k = 2 \frac{\text{lb}}{\text{ft}}$$

(b) The weight is then attached to this same spring. It is released from a position 1 foot below equilibrium with an initial upward velocity of 2 feet per second. Find the equation of motion. (Use $g = 32 \text{ ft/sec}^2$ to determine the mass.)

$$W = mg \Rightarrow m = \frac{W}{g} = \frac{4 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{8} \text{ slug}$$

$$\text{For displacement } x : mx'' + kx = 0 \Rightarrow \frac{1}{8}x'' + 2x = 0$$

$$x'' + 16x = 0, \quad x(0) = 1, \quad x'(0) = -2$$

$$\text{Here, } \omega^2 = 16 \Rightarrow \omega = 4$$

$$x = C_1 \cos 4t + C_2 \sin 4t \quad x' = 4C_1 \sin 4t + 4C_2 \cos 4t$$

$$x(0) = C_1 = 1$$

$$x'(0) = 4C_2 = -2$$

The equation of motion is

$$x = \cos 4t - \frac{1}{2} \sin 4t$$

(3) Find the Fourier series of the function

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ 2, & 0 \leq x < 1 \end{cases}.$$

$$a_0 = \frac{1}{\pi} \int_{-1}^1 f(x) dx = \int_{-1}^0 dx + \int_0^1 2 dx = x \Big|_{-1}^0 + 2x \Big|_0^1 \\ = 0 - (-1) + 2(1 - 0) = 3$$

$$a_n = \frac{1}{\pi} \int_{-1}^1 f(x) \cos(n\pi x) dx = \int_{-1}^0 \cos(n\pi x) dx + 2 \int_0^1 \cos(n\pi x) dx \\ = \frac{1}{n\pi} \sin(n\pi x) \Big|_{-1}^0 + \frac{2}{n\pi} \sin(n\pi x) \Big|_0^1 = 0$$

$$b_n = \frac{1}{\pi} \int_{-1}^1 f(x) \sin(n\pi x) dx = \int_{-1}^0 \sin(n\pi x) dx + 2 \int_0^1 \sin(n\pi x) dx \\ = \frac{-1}{n\pi} \cos(n\pi x) \Big|_{-1}^0 + \frac{-2}{n\pi} \cos(n\pi x) \Big|_0^1 \\ = \frac{-1}{n\pi} [\cos 0 - \cos(-n\pi)] - \frac{2}{n\pi} [\cos(n\pi) - \cos 0] \\ = \frac{-1}{n\pi} (1 - (-1)^n) - \frac{2}{n\pi} ((-1)^n - 1) = \frac{1}{n\pi} (1 - (-1)^n)$$

$$\boxed{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin(n\pi x)}$$

(4) Find the half range cosine series of the function $f(x) = x$, for $0 < x < \frac{\pi}{2}$.

Here $P = \frac{\pi}{2}$ so that $\frac{n\pi x}{P} = 2nx$

$$\text{Cosine : } a_0 = \frac{2}{\pi/2} \int_0^{\pi/2} f(x) dx = \frac{4}{\pi} \int_0^{\pi/2} x dx = \frac{4}{\pi} \left(\frac{x^2}{2} \right)_0^{\pi/2} = \frac{4}{\pi} \left(\frac{\pi^2}{8} \right) = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{2}{\pi/2} \int_0^{\pi/2} f(x) \cos\left(\frac{n\pi x}{\pi/2}\right) dx = \frac{4}{\pi} \int_0^{\pi/2} x \cos(2nx) dx = \\ &= \frac{4}{\pi} \left[\frac{x}{2n} \sin(2nx) \right]_0^{\pi/2} - \frac{1}{2n} \int_0^{\pi/2} \sin(2nx) dx = \\ &= \frac{4}{\pi} \left[\frac{1}{(2n)^2} \cos(2nx) \right]_0^{\pi/2} = \frac{1}{n^2 \pi} [\cos(n\pi) - 1] = \frac{(-1)^n - 1}{n^2 \pi} \end{aligned}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi} \cos(2nx)$$

$$\text{Since : } b_n = \frac{2}{\pi/2} \int_0^{\pi/2} f(x) \sin\left(\frac{n\pi x}{\pi/2}\right) dx = \frac{4}{\pi} \int_0^{\pi/2} x \sin(2nx) dx$$

$$\begin{aligned} &= \frac{4}{\pi} \left[-\frac{x}{2n} \cos(2nx) \right]_0^{\pi/2} + \frac{1}{2n} \int_0^{\pi/2} \cos(2nx) dx \\ &= \frac{4}{\pi} \left[-\frac{\pi/2}{2n} \cos(n\pi) - 0 \right] = -\frac{1}{n} (-1)^n = \frac{(-1)^{n+1}}{n} \end{aligned}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx)$$

(5) Use the Laplace transform to solve the initial value problem. Laplace transforms must be used to receive credit.

$$y' + y = U(t - 1), \quad y(0) = 1$$

$$\mathcal{L}\{y' + y\} = \mathcal{L}\{U(t-1)\}$$

$$sY(s) - y(0) + Y(s) = \frac{e^{-s}}{s}$$

$$(s+1)Y(s) = \frac{e^{-s}}{s} + 1$$

$$Y(s) = \frac{e^{-s}}{s(s+1)} + \frac{1}{s+1}$$

$$\text{Decomp: } \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow 1 = A(s+1) + Bs$$

set $s=0$, $A=1$
 $s=-1$, $B=-1$

$$Y(s) = \frac{e^{-s}}{s} - \frac{e^{-s}}{s+1} + \frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = U(t-1) - e^{-(t-1)}U(t-1) + e^{-t}$$

(6) Find the general solution of the nonhomogeneous equation using any applicable method.

$$y'' - 9y = 18 - 4e^x$$

Find y_c : $m^2 - 9 = 0 \Rightarrow m = \pm 3$

so $y_c = C_1 e^{3x} + C_2 e^{-3x}$

Find y_p : Using Undetermined Coefficients

$$y_p = A + Be^x$$

$$y_p = Be^x$$

$$y_p'' = Be^x$$

$$y_p'' - 9y_p = 18 - 4e^x$$

$$Be^x - 9(A + Be^x) = 18 - 4e^x$$

$$\begin{aligned} -9A - 8Be^x &= 18 - 4e^x \Rightarrow & -9A &= 18 & A &= -2 \\ -8B &= -4 & B &= \frac{1}{2} \end{aligned}$$

$$y_p = -2 + \frac{1}{2}e^x$$

The general solution is

$$y = C_1 e^{3x} + C_2 e^{-3x} - 2 + \frac{1}{2}e^x$$

(7) One solution of the homogeneous equation is given. Find a second linearly independent solution, and determine the general solution of the equation.

$$x^2y'' - 3xy' + 4y = 0, \quad x > 0, \quad y_1(x) = x^2$$

In standard form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$P(x) = \frac{-3}{x}, \quad - \int P(x) dx = \int \frac{3}{x} dx = 3 \ln x = \ln x^3$$

Using reduction of order

$$\begin{aligned} u &= \int \frac{e^{-\int P(x) dx}}{x^2} dx = \int \frac{e^{-\ln x^3}}{(x^2)^2} dx = \int \frac{x^3}{x^4} dx \\ &= \int \frac{1}{x} dx = \ln x \end{aligned}$$

$$\therefore y_2 = uy_1 = (\ln x)x^2 = x^2 \ln x$$

The general solution is

$$y = C_1 x^2 + C_2 x^2 \ln x$$