Final Exam Math 2254 sec. 004

Spring 2014

Name: (4 points) w/ Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	

INSTRUCTIONS: There are 8 problems worth 12 points each. **No** use of notes, books, or calculators is allowed. Cell phones, tablets, or other app. running, internet access devices must be put away. **Illicit use of notes, a calculator, a phone or app. running device will result in a grade of zero for this exam and may result in removal from this class.** To receive full credit, you must follow the directions given and clearly justify your answer. Answers without supporting work will not be considered for credit.

(1) Evaluate the derivative of each function.

(a) $f(x) = x^2 \ln(e^x + e^{-x})$

$$f'(x) = 2x \ln(e^x + e^{-x}) + x^2 \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

(b) $g(x) = \sin^{-1}(\sqrt{x})$

$$g'(x) = \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

(c) $h(x) = x^x$

$$\ln h(x) = \ln x^x = x \ln x$$

$$\frac{h'(x)}{h(x)} = \ln x + x \frac{1}{x} = \ln x + 1$$

$$h'(x) = h(x)(\ln x + 1) \Rightarrow h'(x) = x^x (\ln x + 1)$$

(2) Evaluate each indefinite integral.

(a) $\int x \ln x \, dx$

By parts: $u = \ln x \quad du = \frac{1}{x} dx$

$v = \frac{x^2}{2} \quad dv = x \, dx$

$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C$

(b) $\int \frac{\tan^{-1} x}{1 + x^2} \, dx$

$u = \tan^{-1} x \quad du = \frac{1}{1+x^2} dx$

$= \int u \, du$

$= \frac{u^2}{2} + C$

$= \frac{1}{2} (\tan^{-1} x)^2 + C$

(3) Find the equation of the line tangent to the parametric curve at the indicated point. Express your final answer in the form $y = mx + b$.

(a) $x = \ln t, \quad y = \sin(\pi t) + t, \quad$ at $\quad (0, 1)$

$\frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = \pi \cos(\pi t) + 1$

When $x = 0$, $0 = \ln t \Rightarrow \underline{\underline{t = 1}}$

note: $\sin(\pi) + 1 = 1$

The slope $m = \left.\frac{dy}{dx}\right|_{t=1} = \frac{\pi \cos(\pi) + 1}{1/1} = -\pi + 1$

$y - 1 = (-\pi + 1)(x - 0)$

$\Longrightarrow \quad y = (1 - \pi)x + 1$

(b) $x = t^2 + 1, \quad y = t^4, \quad$ at the point where $\quad t = 1$

$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 4t^3 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{4t^3}{2t} = 2t^2$

when $t = 1$, $x = 2$ $\quad y = 1$ and slope $m = 2$

$y - 1 = 2(x - 2)$

$y = 2x - 3$

(4) (a) Determine whether the integral is proper or improper. If it is improper, briefly describe why it is improper.

$$\int_0^1 \frac{dx}{\sqrt[3]{x}}$$

It's improper. The integrand is not defined at zero.

(b) If possible, evaluate the integral. Otherwise, show that it is divergent.

$$\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{t \to 0^+} \int_t^1 x^{-1/3}\, dx$$

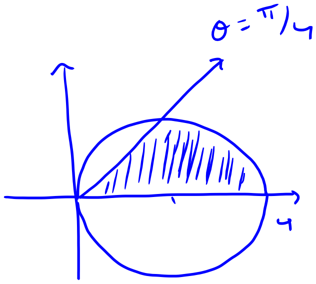
$$= \lim_{t \to 0^+} \left. \frac{x^{2/3}}{2/3} \right|_t^1$$

$$= \lim_{t \to 0^+} \left. \frac{3}{2} x^{2/3} \right|_t^1$$

$$= \lim_{t \to 0^+} \frac{3}{2}(1) - \frac{3}{2}(t)^{2/3}$$

$$= \frac{3}{2} - 0 = \frac{3}{2}$$

(5) Find the area of the region bound by the graph of the polar equation $r = 4\cos\theta$ for $0 \le \theta \le \frac{\pi}{4}$.



Circle of radius 2 centered @ (2,0)

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

$$A = \frac{1}{2}\int_{0}^{\pi/4} (4\cos\theta)^2 \, d\theta$$

$$= \frac{16}{2}\int_{0}^{\pi/4} \cos^2\theta \, d\theta$$

$$= 8\int_{0}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 4\int_{0}^{\pi/4} (1 + \cos 2\theta)\, d\theta$$

$$= 4\left(\theta + \frac{1}{2}\sin 2\theta \right|_{0}^{\pi/4}$$

$$= 4\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - 0 - \frac{1}{2}\sin(0)\right)$$

$$= 4\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

$$= \pi + 2$$

(6) Evaluate each indefinite integral.

(a) $\int \cos^2\theta \sin^3\theta \, d\theta = \int \cos^2\theta \sin^2\theta \sin\theta \, d\theta$

$= \int \cos^2\theta \, (1-\cos^2\theta) \sin\theta \, d\theta$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$= \int (\cos^2\theta - \cos^4\theta) \sin\theta \, d\theta$

$= -\int (u^2 - u^4) \, du$

$= -\frac{u^3}{3} + \frac{u^5}{5} + C$

$= -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C$

(b) $\int 4\sec(2x)\, dx$

$u = 2x \qquad du = 2dx$

$= 2\int \sec u \, du$

$= 2\ln|\sec u + \tan u| + C$

$= 2\ln|\sec(2x) + \tan(2x)| + C$

(7) Determine whether the series converges conditionally, converges absolutely or diverges. Clearly state your conclusion with justification.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$$

The terms are all positive, so

$$\sum_{n=2}^{\infty} \left| \frac{\ln n}{n^4} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n^4}.$$

Direct comparison test:

For all $n \geq 2$ $\quad 0 \leq \ln n \leq n$

so $\quad 0 \leq \frac{\ln n}{n^4} \leq \frac{n}{n^4} = \frac{1}{n^3}$

The series $\sum_{n=2}^{\infty} \frac{1}{n^3}$ is a convergent

p-series w/ $p = 3 > 1$.

Our series w/ smaller terms converges

by direct comparison.

The series converges absolutely.

(8) Determine the function $y = f(x)$ that satisfies the following conditions:

$$\frac{dy}{dx} = \frac{x-5}{(x+1)(x-2)}, \quad \text{and } \; y = 0 \; \text{ when } \; x = 0.$$

$y = \int \frac{dy}{dx}\, dx$

partial fractions:

$$\frac{x-5}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$x-5 = A(x-2) + B(x+1)$$

$x = -1 \quad -6 = -3A \Rightarrow A = 2$

$x = 2 \quad -3 = 3B \Rightarrow B = -1$

$$y = \int \left(\frac{2}{x+1} - \frac{1}{x-2} \right) dx$$

$$= 2\ln|x+1| - \ln|x-2| + C$$

applying the initial condition

$y(0) = 0 = 2\ln|1| - \ln|-2| + C \Rightarrow$

$C = \ln 2 - 2\ln 1 = \ln 2$

$$y = 2\ln|x+1| - \ln|x-2| + \ln 2$$

$$\boxed{y = \ln\left(\frac{2(x+1)^2}{|x-2|} \right)}$$