

Final Exam Math 2306 sec. 5

Fall 2013

Name: (4 points)

Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: _____

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 16 points each. Credit will be awarded for 6 problems. You may eliminate any one problem, or I will count your best 6 out of 7. You may use one $8.5'' \times 11''$ page of your own notes/formulas. **No** use of books, or calculators is allowed. Cell phones, tablets, or other app. running, internet access devices must be put away. **Illicit use of a calculator, a phone or app. running device will result in a grade of zero for this exam and may result in removal from this class.** To receive full credit, you must follow the directions given and clearly justify your answer. Answers without supporting work will not be considered for credit.

(1) Solve the initial value problem using any applicable method.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 4, \quad y(0) = 0, \quad y'(0) = 5$$

Find y_c : $y'' + 2y' + y = 0$ $m^2 + 2m + 1 = 0$
 $(m+1) = 0 \Rightarrow m = -1$ repeated

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

For y_p , use undetermined coefficients with $y_p = A$

$$y_p' = y_p'' = 0$$

$$y_p'' + 2y_p' + y_p = 4 \Rightarrow A = 4$$

$$\text{so } y_p = 4$$

The general solution is $y = c_1 e^{-x} + c_2 x e^{-x} + 4$

Apply the I.C. $y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x}$

$$y(0) = c_1 + 4 = 0 \Rightarrow c_1 = -4$$

$$y'(0) = -c_1 + c_2 = 5 \Rightarrow c_2 = 5 + c_1 = 1$$

The solution to the IVP is

$$y = -4 e^{-x} + x e^{-x} + 4$$

(2) Suppose we wish to use the method of undetermined coefficients to find the particular solution. Determine the **form** of the particular solution. (Note that you are not being asked to find any of the coefficient values.)

(a) $y'' + 6y' + 10y = 4e^{-3x} \sin x$

$$m^2 + 6m + 10 = 0$$

$$m^2 + 6m + 9 + 1 = 0$$

$$(m+3)^2 = -1 \Rightarrow m = -3 \pm i$$

$$y_1 = e^{-3x} \cos x, \quad y_2 = e^{-3x} \sin x$$

$$y_p = (A e^{-3x} \cos x + B e^{-3x} \sin x) x$$

$$y_p = A x e^{-3x} \cos x + B x e^{-3x} \sin x$$

(b) $y'' + 4y' = 2x$

$$m^2 + 4m = 0 \Rightarrow m(m+4) = 0 \quad m = 0 \text{ or } m = -4$$

$$y_1 = 1 \quad y_2 = e^{-4x}$$

$$y_p = (Ax + B) x$$

$$y_p = Ax^2 + Bx$$

(3) Solve the initial value problem using the Laplace transform.

$$y' + 3y = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t \end{cases} \quad y(0) = 2$$

$$= u(t-1)$$

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{u(t-1)\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{e^{-s}}{s} \Rightarrow (s+3)Y(s) = \frac{e^{-s}}{s} + 2$$

$$Y(s) = \frac{e^{-s}}{s(s+3)} + \frac{2}{s+3}$$

$$\text{Decomp: } \frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \Rightarrow 1 = A(s+3) + Bs$$

$$\begin{aligned} \text{set } s=0 & \quad A = \frac{1}{3} \\ s=-3 & \quad B = -\frac{1}{3} \end{aligned}$$

$$Y(s) = \frac{1}{3} \frac{e^{-s}}{s} - \frac{1}{3} \frac{e^{-s}}{s+3} + \frac{2}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}u(t-1) - \frac{1}{3}e^{-3(t-1)}u(t-1) + 2e^{-3t}$$

(4) Find the Fourier series of the given function.

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \leq x < 1 \end{cases}$$

$$a_0 = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_0^1 dx = x \Big|_0^1 = 1$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(x) \cos\left(\frac{n\pi x}{1}\right) dx = \int_0^1 \cos(n\pi x) dx = \frac{1}{n\pi} \sin(n\pi x) \Big|_0^1 = 0$$

$$b_n = \frac{1}{1} \int_{-1}^1 f(x) \sin\left(\frac{n\pi x}{1}\right) dx = \int_0^1 \sin(n\pi x) dx =$$

$$\frac{-1}{n\pi} \cos(n\pi x) \Big|_0^1 = \frac{-1}{n\pi} \cos(n\pi) + \frac{1}{n\pi} \cos 0$$

$$= \frac{1}{n\pi} (1 - (-1)^n)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin(n\pi x)$$

(5) Determine if the given function is even or odd. Find the Fourier series of f . (You may wish to use the symmetry to simplify the process of finding the series.)

$$f(x) = |x| \quad -\pi < x < \pi$$

$$f(-x) = |-x| = |x| = f(x) \quad \text{f is an even function.}$$

The series will not contain sine terms i.e. $b_n = 0$ for all n .

By
Symmetry

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \left(\frac{\pi^2}{2} \right) = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx$$

$$\begin{aligned} u &= x & du &= dx \\ dv &= \cos(nx) dx \\ v &= \frac{1}{n} \sin(nx) \end{aligned}$$

$$= \frac{2}{\pi} \left[\frac{x}{n} \sin(nx) \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin(nx) dx \right]$$

$$= \frac{2}{\pi} \left[\frac{1}{n^2} \cos(nx) \Big|_0^{\pi} \right] = \frac{2}{n^2 \pi} [\cos(n\pi) - \cos 0]$$

$$= \frac{2}{n^2 \pi} ((-1)^n - 1)$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} ((-1)^n - 1) \cos(nx)$$

(6) A 4 kg mass is attached to a spring whose spring constant is 24 N/m. The mass is subjected to friction which produces a damping force numerically equal to 20 times the instantaneous velocity. The mass is released from rest from a position 10 cm (0.1 m) below equilibrium. Determine the equation of motion.

$$m = 4, \quad k = 24, \quad \beta = 20 \quad x = \text{displacement}$$

$$4x'' + 20x' + 24x = 0, \quad x(0) = 0.1, \quad x'(0) = 0$$

$$x'' + 5x' + 6x = 0$$

$$r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0$$

$$r = -2 \quad \text{or} \quad r = -3$$

$$x = c_1 e^{-2t} + c_2 e^{-3t}$$

$$x' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

$$\left. \begin{array}{l} x(0) = c_1 + c_2 = 0.1 \\ x'(0) = -2c_1 - 3c_2 = 0 \end{array} \right\} \Rightarrow \begin{array}{l} 2c_1 + 2c_2 = 0.2 \\ -2c_1 - 3c_2 = 0 \end{array}$$

$$\underline{\hspace{10em}} \quad -c_2 = 0.2 \quad c_2 = -0.2$$

$$c_1 = 0.1 - c_2 = 0.1 + 0.2 = 0.3$$

The equation of motion is

$$x(t) = 0.3 e^{-2t} - 0.2 e^{-3t}$$

(7) Solve each first order differential equation using any applicable method. You may provide an implicit or explicit solution, your choice.

(a) $\frac{dy}{dx} = \frac{y^2}{1+x^2}$ This is separable $\frac{1}{y^2} \frac{dy}{dx} = \frac{1}{1+x^2}$

$$\int y^{-2} dy = \int \frac{dx}{1+x^2} \Rightarrow -\frac{1}{y} = \tan^{-1} x + C$$

$$y = \frac{-1}{C + \tan^{-1} x}$$

(b) $y' - y = e^x \cos(x)$ 1st order linear $P(x) = -1$ $\mu = e^{\int P dx} = e^{-x}$

$$\frac{d}{dx} [e^{-x} y] = e^{-x} \cdot e^x \cos x = \cos x$$

$$\int \frac{d}{dx} [e^{-x} y] dx = \int \cos x dx = \sin x + C$$

$$e^{-x} y = \sin x + C$$

$$y = e^x \sin x + C e^x$$