Final Exam Math 2335 sec. 1

Spring 2014

Name: (4 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems worth 16 points each. You may eliminate any one problem, or I will count your best 6 out of 7. You may use your text book (Atkinson & Han), one 8.5" by 11" page of written notes and a calculator. **NO use of cell phones, tablets or any app running, internet access device may be used. Illicit use of such items will result in a grade of zero on this exam.** To receive full credit, you must follow the directions given and clearly justify your answer.

(1) Determine the Taylor polynomial $p_n(x)$ of degree n with the remainder term $R_n(x)$ for the function

$$f(x) = 3^x \quad \text{centered at} \quad a = 1.$$

$f(x) = 3^x$

$f'(x) = 3^x (\ln 3)$

$f''(x) = 3^x (\ln 3)^2$

$f^{(3)}(x) = 3^x (\ln 3)^3$

$\vdots$

$f^{(n)}(x) = 3^x (\ln 3)^n$

$f(a) = f(1) = 3$

so

$f^{(n)}(a) = 3 (\ln 3)^n$

And

$f^{(n+1)}(c) = 3^c (\ln 3)^{n+1}$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

for some c between x and a

$$P_n(x) = 3 + 3\ln 3\,(x-1) + \frac{3(\ln 3)^2}{2!}(x-1)^2 + \dots + \frac{3(\ln 3)^n}{n!}(x-1)^n$$

and

$$R_n(x) = \frac{(x-1)^{n+1}}{(n+1)!} 3^c (\ln 3)^{n+1}$$

for some c between x and 1.

Recall $\frac{d}{dx}u^x = u^x \ln(u)$ for any positive constant u.

(2) Find an approximation or a rearrangement of each expression to avoid loss of significance. Use any applicable combination of algebra, function identities, and Taylor polynomials.

(a) $\dfrac{e^{3x}-1}{x}$ for $x \approx 0$

$$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots + \frac{(3x)^n}{n!} + \frac{x^{n+1}}{(n+1)!} 3^{n+1} e^{3c}$$

for some c between x and 0.

$$\frac{e^{3x}-1}{x} = 3 + \frac{9x}{2} + \frac{27x^2}{6} + \cdots + \frac{3^n x^{n-1}}{n!} + \frac{3^{n+1} x^n}{(n+1)!} e^{3c}$$

(b) $\dfrac{\sqrt{x^2+7}-4}{x-3}$ for $x \approx 3$

$$\frac{\sqrt{x^2+7}-4}{x-3} = \frac{\sqrt{x^2+7}-4}{x-3} \cdot \frac{(\sqrt{x^2+7}+4)}{(\sqrt{x^2+7}+4)} = \frac{x^2+7-16}{(x-3)(\sqrt{x^2+7}+4)}$$

$$= \frac{(x-3)(x+3)}{(x-3)\cdot(\sqrt{x^2+7}+4)} = \frac{x+3}{\sqrt{x^2+7}+4}$$

(3) Approximate the value of the integral. Give four digits to the right of the decimal.

$$\int_{-1}^{1} \ln(x^2+1)\,dx$$

(a) Using S_4.

$h = \frac{1-(-1)}{4} = \frac{1}{2}$ $\quad x_0 = -1,\ x_1 = -\frac{1}{2},\ x_2 = 0,\ x_3 = \frac{1}{2},\ x_4 = 1$

$$S_4 = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{1}{6}\left[\ln 2 + 4\ln\left(\tfrac{5}{4}\right) + 2\ln 1 + 4\ln\left(\tfrac{5}{4}\right) + \ln 2\right]$$

$$\doteq 0.5286$$

(b) Using Gaussian quadrature I_2.

$$I_2(f) = 1\, f\left(\tfrac{-1}{\sqrt{3}}\right) + 1 \cdot f\left(\tfrac{1}{\sqrt{3}}\right)$$

$$= \ln\left(1 + \tfrac{1}{3}\right) + \ln\left(1 + \tfrac{1}{3}\right)$$

$$\doteq 0.5754$$

(4) Use the method of undetermined coefficients to find an approximation formula $D_h f(x)$ to the first derivative of the form

$$f'(x) \approx D_h f(x) = Af(x+2h) + Bf(x+h)$$

$$A f(x+2h) = A f(x) + A\, 2h f'(x) + A \frac{(2h)^2}{2!} f''(x) + A \frac{(2h)^3}{3!} f'''(x) + \dots$$

$$B f(x+h) = B f(x) + B h f'(x) + B \frac{h^2}{2!} f''(x) + B \frac{h^3}{3!} f'''(x) + \dots$$

$$A f(x+2h) + B f(x+h) = (A+B) f(x) + (2hA + Bh) f'(x)$$

$$+ (2h^2 A + \tfrac{1}{2} h^2 B) f''(x) + \dots$$

We want the coefficient of f to be zero and that of f' to be 1

$$\left.\begin{aligned} A + B &= 0 \\ 2hA + hB &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &B = -A \\ &2hA - hA = 1 \Rightarrow hA = 1 \quad A = \frac{1}{h} \end{aligned}$$

So $A = \frac{1}{h}$, $B = \frac{-1}{h}$

$$D_h f(x) = \frac{1}{h} f(x+2h) - \frac{1}{h} f(x+h)$$

$$= \frac{f(x+2h) - f(x+h)}{h}$$

(5) Approximate the derivative $f'(0)$ for the function $f(x) = \cos(e^x)$. (Round all answers to 4 decimal places.)

(a) Using the forward difference formula with $h = 0.1$.

$$f'(0) \approx D_{0.1} f(0) = \frac{f(0+0.1) - f(0)}{0.1}$$

$$\doteq -0.9132$$

(b) Using the backward difference formula with $h = 0.1$.

$$f'(0) \approx D_{0.1} f(0) = \frac{f(0) - f(0-0.1)}{0.1}$$

$$\doteq -0.7751$$

(c) Using the central difference formula with $h = 0.1$.

$$f'(0) \approx D_{0.1} f(0) = \frac{f(0+0.1) - f(0-0.1)}{2(0.1)}$$

$$\doteq -0.8442$$

(6) We wish to find the value of $\sqrt[3]{5}$.

(a) Define a cubic function $f(x)$ having the desired value as its root.

$$f(x) = x^3 - 5$$

(b) Set up Newton's Method to find the solution. Provide the Newton Iteration formula in the space provided. Using an initial guess of $x_0 = 1$, find the first four iterates. Enter them into the table provided **giving 5 digits to the right of the decimal**.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 5}{3x_n^2}$$

$$= x_n - \frac{1}{3}x_n + \frac{5}{3x_n^2}$$

$$= \frac{1}{3}\left(2x_n + \frac{5}{x_n^2}\right)$$

n	x_n
0	1.00000
1	2.33333
2	1.86168
3	1.72200
4	1.71006

Iteration Formula:

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{5}{x_n^2}\right)$$

(7) (a) Find all possible values of the parameters α and β such that the given function $s(x)$ is a cubic spline on the interval $[0, 2]$.

$$s(x) = \begin{cases} x^3 - x + \alpha, & 0 \le x \le 1 \\ -x^3 + 6x^2 + \beta x - 2, & 1 \le x \le 2 \end{cases}$$

Continuity @ 1 requires

$$1^3 - 1 + \alpha = -1^3 + 6\cdot 1^2 + \beta \cdot 1 - 2$$

$$\alpha = 3 + \beta$$

Continuity of $s'(x)$ @ 1 requires

$$3\cdot 1^2 - 1 = -3\cdot 1^2 + 12\cdot 1 + \beta$$

$$2 = 9 + \beta \Rightarrow \beta = -7$$

Then $\alpha = 3 - 7 = -4$

$$\boxed{\begin{array}{l} \text{So} \\ \alpha = -4 \\ \beta = -7 \end{array}}$$

Note $s''(x) = \begin{cases} 6x, & 0 \le x < 1 \\ -6x + 12, & 1 \le x \le 2 \end{cases}$

$$\lim_{x \to 1^-} s''(x) = \lim_{x \to 1^+} s''(x) = 6 \quad \text{as required}$$

(b) Is $s(x)$ a *natural* cubic spline (why or why not)?

$$s''(0) = 6 \cdot 0 = 0 \quad , \quad s''(2) = -12 + 12 = 0$$

Since the 2^{nd} derivative is zero at both ends, the cubic spline is natural.

This page is provided with Spline formulas. You may tear off this page. You do not have to return this page.

To build the natural cubic spline $s(x)$ to interpolate the points (x_j, y_j), $j = 1, \ldots, n$, define the constants

$$M_j = s''(x_j) \quad \text{and} \quad h_j = x_{j+1} - x_j.$$

Then the values M_j are the solutions to the system of equations:

$$M_1 = 0, \quad M_n = 0, \quad \text{and}$$

$$\frac{h_{j-1}}{6} M_{j-1} + \frac{h_j + h_{j-1}}{3} M_j + \frac{h_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{h_j} - \frac{y_j - y_{j-1}}{h_{j-1}}$$

$$\text{for} \quad j = 2, \ldots, n-1.$$

On each subinterval $[x_j, x_{j+1}]$

$$s(x) = \frac{M_j}{6h_j}(x_{j+1}-x)^3 + \frac{M_{j+1}}{6h_j}(x-x_j)^3 + \frac{y_j}{h_j}(x_{j+1}-x) + \frac{y_{j+1}}{h_j}(x-x_j) - \frac{h_j}{6}\left[M_j(x_{j+1}-x) + M_{j+1}(x-x_j)\right]$$

$$j = 1, \ldots, n-1.$$

Note: For equally spaced points, $h_j = h = \textit{constant}$, the equations for M_j simplify to

$$M_{j-1} + 4M_j + M_{j+1} = \frac{6}{h^2}(y_{j+1} - 2y_j + y_{j-1})$$

$$\text{for} \quad j = 2, \ldots, n-1$$