

# Exam I Math 3260 sec. 55

Spring 2018

Name: \_\_\_\_\_ *Solutions* \_\_\_\_\_

Your signature (required) confirms that you agree to practice academic honesty.

Signature: \_\_\_\_\_

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	

**INSTRUCTIONS:** There are 8 problems; the point values are listed with the problems. You may use a calculator with matrix capabilities. **No wifi enabled device can be used as a calculator.** There are no notes, or books allowed. **Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** To receive full credit, you must clearly justify your answer.

(1) (15 points) Use an augmented matrix with hand calculations (i.e. show the row reduction steps) to find the solution set for the given linear system. **Give your solution in parametric vector form.**

$$\begin{aligned} x_1 - 2x_2 + x_3 + 4x_4 &= 6 \\ 2x_1 - 4x_2 - x_3 + 2x_4 &= 9 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 6 \\ 2 & -4 & -1 & 2 & 9 \end{bmatrix}$$

$$-2R_1 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 6 \\ 0 & 0 & -3 & -6 & -3 \end{bmatrix}$$

$$-\frac{1}{3}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$-R_2 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & 5 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$x_1 = 5 + 2x_2 - 2x_4$$

$$x_3 = 1 - 2x_4$$

$x_2, x_4$  - free

$$\vec{x} = \begin{bmatrix} 5 + 2x_2 - 2x_4 \\ x_2 \\ 1 - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad s, t \text{ in } \mathbb{R}$$

(2) (10 points) Determine all values of the variable  $h$  such that the given matrix is the augmented matrix of a consistent linear system.

(a)  $\begin{bmatrix} 1 & 3 & 4 \\ 3 & h & 5 \end{bmatrix} \xrightarrow{-3R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & h-9 & -7 \end{bmatrix}$

For column 3 to not be a pivot, we need  $h \neq 9$

(b)  $\begin{bmatrix} 1 & 4 & h \\ 3 & 12 & 5 \end{bmatrix} \xrightarrow{-3R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & 4 & h \\ 0 & 0 & 5-3h \end{bmatrix}$

It's consistent if  $5-3h=0 \Rightarrow h = \frac{5}{3}$

(3) (10 points) Write each vector equation in the form of a matrix equation  $Ax = b$ . **Note: you are not being asked to solve anything, just to express the equation in a different form.**

(a)  $x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

(b)  $x_1 \begin{bmatrix} -1 \\ 1 \\ 4 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

(4) (15 points) Evaluate each product  $A\mathbf{x}$  for given  $A$  and  $\mathbf{x}$  if possible. If the product is not defined, briefly state why.

Be specific; calculator responses such as *Dimension*, *DIM MISMATCH*, or *ERROR* will not be considered for credit.

$$(a) \quad A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & -1 \\ 3 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1-8 \\ 2-4-4 \\ 3-6 \\ 1+2+8 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \\ -3 \\ 11 \end{bmatrix}$$

$$(b) \quad A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -3 & 3 & 0 \\ 1 & 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$A\vec{x}$  is undefined.

$A$  has 4 columns, but  $\vec{x}$  has only 3 entries.

$$(c) \quad A = \begin{bmatrix} -2 & 1 \\ 3 & -2 \\ 0 & 3 \\ 1 & -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} -2-1 \\ 3+2 \\ 0-3 \\ 1+5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -3 \\ 6 \end{bmatrix}$$

(5) (20 points) Use the given set of vectors to answer the questions that follow.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}.$$

(a) Is  $\mathbf{v}_3$  in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ ? (Justify)

$$[\vec{v}_1, \vec{v}_2, \vec{v}_3] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No,  $\vec{v}_3$  is not in  $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ .  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is lin. independent.

(b) Determine the vector  $3\mathbf{v}_1 - \mathbf{v}_4$

$$= 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

(c) Is the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  linearly dependent or linearly independent? (Justify)

Dependent. This is 4 vectors in  $\mathbb{R}^3$ .

(d) Let  $\mathbf{b} = (2, -2, 0)$ . Find a linear dependence relation for the set  $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{b}\}$ .

$$[\vec{v}_1, \vec{v}_3, \vec{b}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{So } \vec{b} = -\vec{v}_1 + \vec{v}_3$$

A lin. dep. relation is  $\vec{v}_1 + \vec{b} - \vec{v}_3 = \vec{0}$

(6) (10 points) Find the solution set of the linear system of one equation  $2x_1 + x_2 - 4x_3 = 0$  and express the solution in the form  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  for some vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^3$

$$x_1 = \frac{1}{2}(-x_2 + 4x_3) = -\frac{1}{2}x_2 + 2x_3$$

$x_2, x_3$  - free

$$\vec{x} = \begin{bmatrix} -\frac{1}{2}x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is

$$\text{Span} \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(7) (5 points) Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  be vectors in  $\mathbb{R}^5$ . Find a linear dependence relation to show that the set  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{0}\}$  is linearly dependent. (Here,  $\mathbf{0}$  is the zero vector in  $\mathbb{R}^5$ .)

Note that  $1 \neq 0$ . A lin. dependence relation is

$$0\vec{u} + 0\vec{v} + 0\vec{w} + 1\vec{0} = \vec{0}$$

(8) (15 points) Determine if each statement is true or false. If a statement is false, give an explanation or example to show that it is false.

- (a) If the equation  $Ax = b$  has infinitely many solutions for some nonzero vector  $b$ , then the homogeneous equation  $Ax = 0$  has nontrivial solutions.

True. The solutions of  $A\vec{x} = \vec{b}$  are of the form  $\vec{x} = \vec{p} + \vec{v}_n$ . There must be  $\infty$ -many  $\vec{v}_n$ .

- (b) Any linear system of  $n$  equations in  $n$  variables has at most  $n$  solutions.

False. It has none, one, or infinitely many. It can't have  $n$  ( $n > 1$ ) unless it has  $\infty$ -many.

- (c) The homogeneous equation  $Ax = 0$  can have no solutions, one solution, or infinitely many solutions.

False. It is always consistent. Hence it can't have no solutions.

- (d) If the set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is linearly independent in  $\mathbb{R}^n$ , then  $n < 4$ .

False.  $n \geq 4$ . If  $n < 4$ , it would be lin. dependent. (more vectors than each has entries)

- (e) Let  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$  be an  $m \times n$  matrix. If  $Ax = b$  is consistent for some vector  $b$  in  $\mathbb{R}^m$ , then the set  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, b\}$  is linearly independent.

False.  $A\vec{x} = \vec{b}$  is consistent if  $\vec{b}$  is in  $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$ . Hence the set  $\{\vec{a}_1, \dots, \vec{a}_n, \vec{b}\}$  must be lin. dependent.