Exam I Math 3260 sec. 56

Spring 2018

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	

INSTRUCTIONS: There are 8 problems; the point values are listed with the problems. You may use a calculator with matrix capabilities. **No wifi enabled device can be used as a calculator.** There are no notes, or books allowed. **Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** To receive full credit, you must clearly justify your answer.

(1) (15 points) Use an augmented matrix with hand calculations (i.e. show the row reduction steps) to find the solution set for the given linear system. **Give your solution in parametric vector form.**

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & - & x_3 & + & 2x_4 & = & -9 \\ x_1 & - & 2x_2 & + & x_3 & + & 4x_4 & = & 6 \end{array}$$

$$\begin{bmatrix} 2 & -4 & -1 & 2 & -9 \\ 1 & -2 & 1 & 4 & 6 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & -2 & 1 & 4 & 6 \\ 2 & -4 & -1 & 2 & -9 \end{bmatrix}$$

$$-2R_1 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & -2 & 1 & 4 & 6 \\ 0 & 0 & -3 & -6 & -21 \end{bmatrix}$$

$$-\tfrac{1}{3}R_2 \to R_2 \qquad \begin{bmatrix} 1 & -2 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 & 7 \end{bmatrix}$$

$$-R_2 + R_1 \to R_1 \qquad \begin{bmatrix} 1 & -2 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 & 7 \end{bmatrix}$$

$x_1 = -1 + 2x_2 - 2x_4$
$x_3 = 7 - 2x_4$
x_2, x_4 - free

$$\vec{x} = \begin{bmatrix} -1 + 2x_2 - 2x_4 \\ x_2 \\ 7 - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad s, t \text{ in } \mathbb{R}$$

(2) (10 points) Determine all values of the variable h such that the given matrix is the augmented matrix of a consistent linear system.

(a) $\begin{bmatrix} -1 & 3 & 2 \\ 2 & h & 1 \end{bmatrix}$ $2R_1 + R_2 \to R_2$ $\begin{bmatrix} -1 & 3 & 2 \\ 0 & h+6 & 5 \end{bmatrix}$

For the last column to not be a pivot column, we require $h \neq -6$

(b) $\begin{bmatrix} 1 & -3 & h \\ 4 & -12 & 10 \end{bmatrix}$ $-4R_1 + R_2 \to R_2$ $\begin{bmatrix} 1 & -3 & h \\ 0 & 0 & 10-4h \end{bmatrix}$

It's only consistent if $10 - 4h = 0 \Rightarrow h = \frac{5}{2}$

(3) (10 points) Write each vector equation in the form of a matrix equation $A\mathrm{x} = \mathrm{b}$. **Note: you are not being asked to solve anything, just to express the equation in a different form.**

(a) $x_1 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & 3 & 1 \\ 1 & 0 & 2 & -2 \\ 4 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

(b) $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 4 & 1 & 1 \\ 2 & 3 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

(4) (15 points) Evaluate each product $A\mathbf{x}$ for given A and $\mathbf{x}$ if possible. If the product is not defined, briefly state why.

Be specific; calculator responses such as *Dimension*, *DIM MISMATCH*, or *ERROR* will not be considered for credit.

(a) $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & -1 \\ 3 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 3 \end{bmatrix}$

$A\vec{x}$ is undefined. A has 3 columns, but $\vec{x}$ has 4 entries.

(b) $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -3 & 3 & 0 \\ 1 & 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 3+1-2-1 \\ -3-6 \\ 3+2-2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 2 \end{bmatrix}$$

(c) $A = \begin{bmatrix} -2 & 1 \\ 3 & -2 \\ 0 & 3 \\ 1 & -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 2+1 \\ -3-2 \\ 3 \\ -1-5 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 3 \\ -6 \end{bmatrix}$$

(5) (20 points) Use the given set of vectors to answer the questions that follow.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

(a) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly dependent or linearly independent? (Justify)

Dependent. There are 4 vectors in $\mathbb{R}^3$.

(b) Determine the vector $2\mathbf{v}_1 - \mathbf{v}_4$ $= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

(c) Is $\mathbf{v}_3$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$? (Justify)

$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

No. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. independent.

(d) Let $\mathbf{b} = (2, -2, 0)$. Find a linear dependence relation for the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{b}\}$.

$[\vec{v}_1 \ \vec{v}_2 \ \vec{b}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{b} = -\vec{v}_1 + \vec{v}_2$

A lin. dep. relation is $\vec{v}_1 - \vec{v}_2 + \vec{b} = \vec{0}$

(6) (10 points) Find the solution set of the linear system of one equation $2x_1 - 6x_2 + x_3 = 0$ and express the solution in the form Span$\{\mathbf{u}, \mathbf{v}\}$ for some vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$

$$x_1 = \frac{1}{2}(6x_2 - x_3) = 3x_2 - \frac{1}{2}x_3$$

$$\vec{x} = \begin{bmatrix} 3x_2 - \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

So the solution set is

$$\text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(7) (5 points) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^5$. Find a linear dependence relation to show that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{0}\}$ is linearly dependent. (Here, $\mathbf{0}$ is the zero vector in $\mathbb{R}^5$.)

Using the coefficient 1, since $1 \neq 0$

a lin. dependence relation is

$$0\vec{u} + 0\vec{v} + 0\vec{w} + 1\vec{0} = \vec{0}$$

(8) (15 points) Determine if each statement is true or false. If a statement is false, give an explanation or example to show that it is false.

(a) If the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for some nonzero vector $\mathbf{b}$, then the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

True. $A\vec{x} = \vec{b}$ has solutions $\vec{x} = \vec{p} + \vec{v}_h$. Since there are infinitely many, there are nonzero $\vec{v}_h$.

(b) Any linear system of n equations in n variables has at most n solutions.

False. If it has $n > 1$ solutions, it must have infinitely many.

(c) The homogeneous equation $A\mathbf{x} = \mathbf{0}$ can have no solutions, one solution, or infinitely many solutions.

False. It is always consistent, hence it cannot have no solutions.

(d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent in $\mathbb{R}^n$, then $n < 4$.

False. $n \geq 4$. If $n < 4$, there would be more vectors than each has entries.

(e) Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ be an $m \times n$ matrix. If $A\mathbf{x} = \mathbf{b}$ is consistent for some vector $\mathbf{b}$ in $\mathbb{R}^m$, then the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{b}\}$ is linearly independent.

False. $A\vec{x} = \vec{b}$ is consistent means $\vec{b}$ is in Span$\{\vec{a}_1, \ldots, \vec{a}_n\}$. This makes $\{\vec{a}_1, \ldots, \vec{b}\}$ linearly dependent.