Exam I Math 3260 sec. 58

Fall 2017

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	

INSTRUCTIONS: There are 8 problems; the point values are listed with the problems. You may use a calculator with matrix capabilities. **No wifi enabled device can be used as a calculator.** There are no notes, or books allowed. **Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** To receive full credit, you must clearly justify your answer.

(1) (15 points) Solve the linear system by using row reduction on the associated augmented matrix. **Show each step in the row reduction process—i.e. do this one by hand and show all of your work.** Give your answer in parametric or vector parametric form, your choice.

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 3x_1 & + & 5x_2 & + & 3x_3 & = & 4 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 5 & 3 & 4 \\ 2 & 1 & 1 & 4 \end{bmatrix}$$

$-3R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$$\begin{array}{cccc} -3 & -6 & -3 & -3 \\ 3 & 5 & 3 & 4 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & -2 & -2 \\ 2 & 1 & 1 & 4 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

$-R_2 \to R_2$

then

$3R_2 + R_3 \to R_3$

$$\begin{array}{cccc} 0 & 3 & 0 & -3 \\ 0 & -3 & -1 & 2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$-R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$-R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$-2R_2 + R_1 \to R_1$

$$\left.\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right\} \Rightarrow$$

$x_1 = 2$

$x_2 = -1$

$x_3 = 1$

In vector parametric form $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(2) (15 points) The matrices A and B shown below are row equivalent. Use this to find the solution set of the given system of equations.

$$A = \begin{bmatrix} -2 & 2 & -3 & -2 & -8 \\ 3 & -3 & 3 & 1 & 10 \\ 2 & -2 & 2 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} -2x_1 & + & 2x_2 & - & 3x_3 & - & 2x_4 & = & -8 \\ 3x_1 & - & 3x_2 & + & 3x_3 & + & x_4 & = & 10 \\ 2x_1 & - & 2x_2 & + & 2x_3 & & & = & 4 \end{array}$$

From B

$x_1 = 6 + x_2$

x_2 - free

$x_3 = -4$

$x_4 = 4$

In vector parametric form

$$\vec{x} = \begin{bmatrix} 6 \\ 0 \\ -4 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad t \text{ in } \mathbb{R}$$

(3) (15 points) Find the standard matrix for the linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (x_4, -x_3, x_1)$$

$T(1,0,0,0) = (0,0,1)$

$T(0,1,0,0) = (0,0,0)$

$T(0,0,1,0) = (0,-1,0)$

$T(0,0,0,1) = (1,0,0)$

The standard matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(4) (5 points) Suppose the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has the two nontrivial solutions $\mathbf{u}$ and $\mathbf{v}$. Show that every vector in Span$\{\mathbf{u}, \mathbf{v}\}$ also solves the homogeneous equation.

Suppose $\vec{x}$ is in Span$\{\vec{u}, \vec{v}\}$. Then $\vec{x} = c_1\vec{u} + c_2\vec{v}$ for some scalars c_1 and c_2. Using properties of Matrix multiplication

$$A\vec{x} = A(c_1\vec{u} + c_2\vec{v})$$
$$= c_1 A\vec{u} + c_2 A\vec{v}$$
$$= c_1\vec{0} + c_2\vec{0} = \vec{0}$$

Hence $\vec{x}$ solves the homogeneous equation.

(5) (5 points) Suppose that the vector $\mathbf{u}_1$ solves the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}_1$ and that the vector $\mathbf{u}_2$ solves the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}_2$. Show that the vector $\mathbf{u}_1 + \mathbf{u}_2$ solves the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2$.

Again using properties of matrix multiplication

$$A(\vec{u}_1 + \vec{u}_2) = A\vec{u}_1 + A\vec{u}_2$$
$$= \vec{b}_1 + \vec{b}_2 \quad \text{as expected.}$$

(6) (15 points) Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v}_3\}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent or linearly independent? (Justify)

They are lin. dependent. Note $\vec{v}_2 = -2\vec{v}_1$.

So $2\vec{v}_1 + \vec{v}_2 = \vec{0}$.

(b) Is the vector $\mathbf{b} = (0, 0, 1)$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$? (Justify)

No. If $\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2$ then $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \vdots \\ c_1 0 + c_2 0 \end{bmatrix}$

which implies that $1 = 0$.

This is false.

(c) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent or linearly independent? (Justify)

Dependent. From part (a),

$$2\vec{v}_1 + v_2 + 0\vec{v}_3 = \vec{0}$$

Is a lin. dependence relation.

(7) (20 points) Use the given matrices to evaluate each expression. If an expresion does not exist, state why it does not exist.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 3 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}$$

(a) $3C$

$$= \begin{bmatrix} 0 & 3 & -3 \\ -3 & 6 & 9 \end{bmatrix}$$

(b) AC $= \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 1 \\ -2 & 6 & 4 \end{bmatrix}$

(c) CA Undefined # rows of A $\neq$ # columns of C.

(d) Ax where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Undefined. $A\vec{x}$ requires $\vec{x}$ in $\mathbb{R}^2$

(e) $C^T + B$ $= \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 4 \\ 2 & -1 \end{bmatrix}$

(8) (10 points) The homogeneous equation $A\mathbf{x} = \mathbf{0}$ is always consistent because it always has the trivial solution. Let

$$A = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix}, \quad \text{for some real numbers } b, c, \text{ and } d.$$

Determine whether the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has one solution or infinitely many solutions

(a) If $d = bc - 1$.

$$\begin{bmatrix} 1 & b & 0 \\ c & d & 0 \end{bmatrix} \quad -cR_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & b & 0 \\ 0 & d-cb & 0 \end{bmatrix}$$

If $d = bc - 1$, this is $\begin{bmatrix} 1 & b & 0 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow \quad x_1 = x_2 = 0$

Only one solution, the trivial one, would exist.

(b) If $d = bc$.

In this case, the above is

$$\begin{bmatrix} 1 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The system would have infinitely many solutions.