Name: ________________________

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ________________________

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INSTRUCTIONS: There are 8 problems; the point values are listed with the problems. You may use a calculator with matrix capabilities. No wifi enabled device can be used as a calculator. There are no notes, or books allowed. Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. To receive full credit, you must clearly justify your answer.
(1) (15 points) Solve the linear system by using row reduction on the associated augmented matrix. 
Show each step in the row reduction process—i.e. do this one by hand and show all of your work. Give your answer in parametric or vector parametric form, your choice.

\[
x_1 + 2x_2 + x_3 = 1 \\
3x_1 + 5x_2 + 3x_3 = 4 \\
2x_1 + x_2 + x_3 = 4
\]

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & -1 & 0 \\
0 & -3 & -1
\end{bmatrix}
\]

- \( R_2 \rightarrow R_2 \)
- \( R_3 \rightarrow R_3 \)

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

- \( R_3 \rightarrow \) 

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- \( R_1 + R_1 \rightarrow R_1 \)

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\( x_1 = 2 \) 
\( x_2 = -1 \) 
\( x_3 = 1 \)

In vector parametric form, \( \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \)
(2) (15 points) The matrices $A$ and $B$ shown below are row equivalent. Use this to find the solution set of the given system of equations.

$$A = \begin{bmatrix} -2 & 2 & -3 & -2 & -8 \\ 3 & -3 & 3 & 1 & 10 \\ 2 & -2 & 2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{align*}
-2x_1 + 2x_2 - 3x_3 - 2x_4 &= -8 \\
3x_1 - 3x_2 + 3x_3 + x_4 &= 10 \\
2x_1 - 2x_2 + 2x_3 &= 4
\end{align*}$$

From $\begin{bmatrix} 13 \end{bmatrix}$

$$\begin{align*}
x_1 &= 6 + x_2 \\
x_2 &= 4x_2 \\
x_3 &= x_3 \\
x_4 &= x_4
\end{align*}$$

In vector parametric form

$$x = \begin{bmatrix} 6 \\ 0 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

(3) (15 points) Find the standard matrix for the linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (x_4, -x_3, x_1)$$

$$\begin{align*}
T(1, 0, 0, 0) &= (0, 0, 1) \\
T(0, 1, 0, 0) &= (0, 0, 0) \\
T(0, 0, 1, 0) &= (0, -1, 0) \\
T(0, 0, 0, 1) &= (1, 0, 0)
\end{align*}$$

The standard matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
(4) (5 points) Suppose the homogeneous equation $Ax = 0$ has the two nontrivial solutions $u$ and $v$. Show that every vector in $\text{Span}\{u, v\}$ also solves the homogeneous equation.

Suppose $\tilde{x}$ is in $\text{Span}\{u, v\}$. Then $\tilde{x} = c_1 u + c_2 v$ for some scalars $c_1$ and $c_2$. Using properties of matrix multiplication,

\[ A\tilde{x} = A(c_1 u + c_2 v) = c_1 A u + c_2 A v = \tilde{0} + \tilde{0} = \tilde{0}, \]

Hence $\tilde{x}$ solves the homogeneous equation.

(5) (5 points) Suppose that the vector $u_1$ solves the nonhomogeneous equation $Ax = b_1$ and that the vector $u_2$ solves the nonhomogeneous equation $Ax = b_2$. Show that the vector $u_1 + u_2$ solves the nonhomogeneous equation $Ax = b_1 + b_2$.

Again using properties of matrix multiplication,

\[ A(u_1 + u_2) = A u_1 + A u_2 = \tilde{b}_1 + \tilde{b}_2 \text{ as expected.} \]
(6) (15 points) Consider the set of vectors \( \{v_1, v_2, v_3\} \).

\[
\begin{align*}
v_1 &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

(a) Is the set \( \{v_1, v_2\} \) linearly dependent or linearly independent? (Justify)

They are linear dependent. Note \( v_2 = -2v_1 \).
So \( 2v_1 + v_2 = \overrightarrow{0} \).

(b) Is the vector \( b = (0, 0, 1) \) in \( \text{Span}\{v_1, v_2\} \)? (Justify)

No. If \( b = c_1 v_1 + c_2 v_2 \) then \( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \) which implies that \( 1 = 0 \).
This is false.

(c) Is the set \( \{v_1, v_2, v_3\} \) linearly dependent or linearly independent? (Justify)

Dependent. From part (a),
\[
2v_1 + v_2 + 0v_3 = \overrightarrow{0}
\]
Is a linear dependence relation.
(7) (20 points) Use the given matrices to evaluate each expression. If an expression does not exist, state why it does not exist.

\[ A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 3 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix} \]

(a) \( 3C \)
\[ = \begin{bmatrix} 0 & 3 & -3 \\ -3 & 6 & 9 \end{bmatrix} \]

(b) \( AC \)
\[ = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 1 \\ -2 & 6 & 4 \end{bmatrix} \]

(c) \( CA \) undefined \( \# \) rows of \( A \) \( \neq \) \( \# \) columns of \( C \).

(d) \( Ax \) where \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) undefined. \( Ax \) requires \( x \) in \( \mathbb{R}^3 \).

(e) \( C^T + B \)
\[ = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 4 \\ 2 & -1 \end{bmatrix} \]
(8) (10 points) The homogeneous equation $Ax = 0$ is always consistent because it always has the trivial solution. Let

$$A = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix}, \quad \text{for some real numbers } b, c, \text{ and } d.$$

Determine whether the homogeneous equation $Ax = 0$ has one solution or infinitely many solutions

(a) If $d = bc - 1$.

$$\begin{bmatrix} 1 & b & 0 \\ c & d & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & b & 0 \\ 0 & d - cb & 0 \end{bmatrix}$$

If $d = bc - 1$, then $\begin{bmatrix} 1 & b & 0 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow x_1 = x_2 = 0$

Only one solution, the trivial one, would exist.

(b) If $d = bc$.

In this case, the above is

$$\begin{bmatrix} 1 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The system would have infinitely many solutions.