Exam 2 Math 3260 sec. 57

Fall 2017

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	

INSTRUCTIONS: There are 8 problems; the point values are listed with the problems. You may use a calculator with matrix capabilities. **No wifi enabled device can be used as a calculator.** There are no notes, or books allowed. **Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** To receive full credit, you must clearly justify your answer.

(1) (10 points) Find all of the values of x such that the matrix A is singular—i.e. not invertible. (Hint: You may wish to consider the determinant.)

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & x-1 & 2 \\ 0 & 3 & x-2 \end{bmatrix}$$

$\det(A) = x\big((x-1)(x-2)-6\big)$

$= x(x^2-3x+2-6)$

$= x(x^2-3x-4) = x(x-4)(x+1)$

A is singular if $\det(A) = 0$.

$\det(A) = 0 \Rightarrow 0 = x(x-4)(x+1) \Rightarrow x = 0, 4,$ or -1.

A is singular if $x = 0, 4,$ or -1.

(2) (15 points) For the given matrix B, compute the cofactors C_{11}, C_{21}, C_{31}. **Note the indices of the requested cofactors.**

$$B = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & 3 \\ 2 & 2 & -2 \end{bmatrix}$$

$C_{11} = (-1)^2 \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = -2-6 = -8$

$C_{21} = (-1)^3 \begin{vmatrix} 4 & -1 \\ 2 & -2 \end{vmatrix} = -(-8+2) = 6$

$C_{31} = (-1)^4 \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix} = (12+1) = 13$

$C_{11} =$ -8 $\qquad C_{21} =$ 6 $\qquad C_{31} =$ 13

(3) (10 points) Use coordinate vectors to determine if the given polynomials are linearly dependent or independent in $\mathbb{P}_3$. Use the elementary basis in $\mathbb{P}_3$. If they are dependent, clearly identify a linear dependence relation.

$$\mathbf{p}_1 = t + t^2 - t^3, \quad \mathbf{p}_2 = 2 + 3t - 2t^2, \quad \mathbf{p}_3 = 4 + 5t - 5t^2 + t^3, \quad \mathbf{p}_4 = 1 + t + 2t^3$$

$$[\vec{p}_1]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad [\vec{p}_2]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 0 \end{bmatrix} \quad [\vec{p}_3]_{\mathcal{E}} = \begin{bmatrix} 4 \\ 5 \\ -5 \\ 1 \end{bmatrix} \quad [\vec{p}_4]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 3 & 5 & 1 \\ 1 & -2 & -5 & 0 \\ -1 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ non pivot column

They are linearly dependent.

From the matrix, $\vec{p}_3 = -\vec{p}_1 + 2\vec{p}_2$

$$\vec{p}_1 - 2\vec{p}_2 + \vec{p}_3 = \vec{0}$$

is a linear dependence relation.

(4) (25 points) Find bases for each of the row space, column space, and null space of the given matrix. State the rank and the nullity of the matrix.

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 & 6 \\ 0 & 4 & 4 & 0 & 4 \\ 3 & -1 & -7 & 1 & 1 \\ 2 & 1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_3$
$x_2 = -x_3 - x_5$
x_3 free
$x_4 = -2x_5$
x_5 free

Pivot Columns 1, 2, 4

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 6 \end{bmatrix} \right\}$$

These are bases

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

For $\bar{x}$ in Nul A $\quad \bar{x} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

this is a basis.

Rank A = 3 $\qquad$ Nullity of A = 2

(5) (5 points) Suppose that A, B and P are 30×30 matrices and that P is invertible[1].
If $B = P^{-1}AP$, show that det(B) = det(A).

$B = P^{-1}AP \Rightarrow \det(B) = \det(P^{-1}AP)$

$= \det(P^{-1})\det(A)\det(P)$

Since scalar multiplication commutes

$\det(B) = \det(P^{-1})\det(P)\det(A)$

$= 1 \cdot \det(A)$

$= \det(A)$ as required.

(6) (20 points) Solve the system using a matrix inverse.

$$\begin{array}{rcrcl} 3x_1 & + & 2x_2 & = & 4 \\ 2x_1 & - & 5x_2 & = & 7 \end{array}$$

$\begin{bmatrix} 3 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ $A = \begin{bmatrix} 3 & 2 \\ 2 & -5 \end{bmatrix}$ $\det(A) = -15-4$

$= -19$

$A^{-1} = \frac{-1}{19}\begin{bmatrix} -5 & -2 \\ -2 & 3 \end{bmatrix}$ $A^{-1}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{-1}{19}\begin{bmatrix} -20-14 \\ -8+21 \end{bmatrix} = \frac{-1}{19}\begin{bmatrix} -34 \\ 13 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{34}{19} \\ \frac{-13}{19} \end{bmatrix}$

[1] You may use the established result that det(P^{-1})det$(P) = 1$.

(7) (10 points) Suppose the matrices A and B are row equivalent, and B was obtained from A by doing the following sequence of row operations to A:

1. $R_1 \leftrightarrow R_3$ (-1)
2. $-2R_1 + R_2 \longrightarrow R_2$ no change
3. $\frac{1}{2}R_2 \longrightarrow R_2$ $\frac{1}{2}$ factor
4. $3R_2 + R_3 \longrightarrow R_3$ no change
5. $3R_4 \longrightarrow R_4$ 3 factor

$\det(B) = (-1) \cdot \frac{1}{2} \cdot 3 \det(A)$

(a) If $\det(A) = 6$, determine the value of $\det(B)$.

$\det B = \frac{-3}{2} \det(A) = \frac{-3}{2}(6) = -9$

(b) If instead $\det(B) = -15$, what is the value of $\det(A)$?

$-15 = \frac{-3}{2} \det(A) \Rightarrow \det(A) = -\frac{2}{3}(-15) = 10$

(8) (5 points) Either show that the given set is a subspace of $\mathbb{P}_2$ by finding a spanning set, or show that it is not a subspace of $\mathbb{P}_2$ by demonstrating that it violates one of the necessary properties of subspaces.

$$\{\mathbf{p}(t) \in \mathbb{P}_2 \;:\; \mathbf{p}(0) = \mathbf{p}(1)\}$$

If $\vec{p} = p_0 + p_1 t + p_2 t^2$ and $\vec{p}(0) = \vec{p}(1)$, then

$\vec{p}(0) = p_0 = \vec{p}(1) = p_0 + p_1 + p_2$

$\Rightarrow p_1 + p_2 = 0 \quad p_1 = -p_2$

So $\vec{p} = p_0 + p_1(t - t^2)$

It is a subspace with spanning set

$\{1, t - t^2\}$.