Exam III Math 3260 sec. 55

Spring 2018

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	

INSTRUCTIONS: There are 7 problems; the point values are listed with the problems. You may use a calculator with matrix capabilities. **No wifi enabled device can be used as a calculator.** There are no notes, or books allowed. **Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** To receive full credit, you must clearly justify your answer.

(1) (15 points) Find an **orthonormal** basis for the column space of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$\vec{u}_1 \quad \vec{u}_2$

Orthogonal : $\vec{v}_1 , \vec{v}_2$

$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$

Orthogonal $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$

$\|\vec{v}_1\| = \sqrt{1+1} = \sqrt{2}$, $\quad \|\vec{v}_2\| = \sqrt{\frac{1}{4}+1+\frac{1}{4}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$

Orthonormal

$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$

(2) (10 points) The given set $\mathcal{B}$ is a basis for $\mathbb{R}^2$. Determine the change of coordinates matrix $P_{\mathcal{B}}$ and its inverse (use the order presented here). Then use this to find the indicated coordinate vectors.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

$P_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \qquad \det(P_{\mathcal{B}}) = 4 \qquad P_{\mathcal{B}}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

Determine $[\mathbf{x}]_{\mathcal{B}}$ for (a) $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, (b) $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x} = \frac{1}{4} \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \frac{1}{4} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x} = \frac{1}{4} \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$= \frac{1}{4} \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(3) (15 points) Determine the dimension of each vector space described.

(a) The space $\mathbb{P}_4$ of polynomials of degree at most 4.

$\dim(\mathbb{P}_4) = 5$

(b) The subspace of $\mathbb{R}^6$ of vectors of the form $(0, x_2, x_3, x_4, x_5, 0)$.

The dimension is 4

(c) The null space of a 7×9 matrix A if $\operatorname{rank}(A) = 6$.

nullity $+ 6 = 9 \Rightarrow$ nullity $= 3$

(4) (10 points) Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$.

(a) Determine the norm $\|\mathbf{v}\|$.

$\|\vec{v}\| = \sqrt{1+4+4+1} = \sqrt{10}$

(b) Find a unit vector parallel to $\mathbf{v}$. (Call the new vector $\mathbf{u}$.)

$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 2/\sqrt{10} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$

(5) (25 points) Find bases for each of Col$A$, Row$A$, and Nul$A$. State the rank and the nullity of A where

$$A = \begin{bmatrix} 1 & 3 & 2 & 7 \\ 2 & 6 & 4 & 14 \\ 1 & 3 & 1 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{x}$ in Nul A

$x_1 = -3x_2 + x_4$

$x_3 = -4x_4$

x_2, x_4 free

(Be sure to write any calculator output on this page to show that upon which your conclusions are based.)

Bases are:

$$\text{Nul } A : \text{Span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

$$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}$$

$$\text{Row } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$$

rank $A = 2$, Nullity $(A) = 2$

(6) (15 points) Find the orthogonal projection of $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ onto the line $L = \text{Span}\left\{ \overset{\vec{u}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\}$, and determine the distance from $\mathbf{y}$ to L. (Hint: Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is the orthogonal projection.)

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{6}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

The orthogonal projection is $\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$.

$$\vec{y} - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

The distance from $\vec{y}$ to L is

$$\left\| \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} \right\| = \sqrt{4+4+4} = \sqrt{12}$$

(7) (10 points) Let $A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$. One of the eigenvalues of A is $\lambda = 3$. Find a basis for the associated eigenspace.

$$A\vec{x} = 3\vec{x} \Rightarrow \begin{bmatrix} 2-3 & 1 \\ -2 & 5-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 - \text{free} \end{matrix}$$

So $\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

The Gram-Schmidt Process: Given a set of basis vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ in some inner product space $\mathbf{V}$, we can obtain an orthonormal set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ that spans the same subspace. First, we obtain the orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ where

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_n &= \mathbf{u}_n - \sum_{k=1}^{n-1} \frac{\mathbf{u}_n \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k}\mathbf{v}_k
\end{aligned}$$

Finally, we obtain the orthonormal set by dividing each $\mathbf{v}_i$ by its norm. That is, for $i = 1, .., n$ set

$$\mathbf{w}_i = \frac{1}{\sqrt{\mathbf{v}_i \cdot \mathbf{v}_i}}\,\mathbf{v}_i.$$