Exam 3 Math 3260 sec. 58

Fall 2017

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems; the point values are listed with the problems. You may use a calculator with matrix capabilities. **No wifi enabled device can be used as a calculator.** There are no notes, or books allowed. **Illicit use of a smart phone, tablet, device that runs apps, or hand written notes will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** To receive full credit, you must clearly justify your answer.

(1) (15 points) Use Gram Scmidt to find an **orthogonal** basis for $\mathrm{Col}(A)$ where $A = \begin{array}{c} \begin{bmatrix} 1 & -2 & 4 \\ 1 & 2 & 2 \\ 0 & 1 & 4 \end{bmatrix} \\ \vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \end{array}$.

(Answers without supporting work will not be considered.)

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 \cdot \vec{v}_1 = -2 + 2 = 0$$

$$\|\vec{v}_1\|^2 = 2$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} - 0 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{u}_3 \cdot \vec{v}_1 = 4 + 2 = 6$$

$$\vec{u}_3 \cdot \vec{v}_2 = -8 + 4 + 4 = 0$$

$$\|\vec{v}_2\|^2 = 4 + 4 + 1 = 9$$

$$\vec{v}_3 = \vec{u}_0 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} - \frac{6}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 0$$

$$= \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

An orthogonal basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

(2) (20 points) Consider the vectors in $\mathbb{R}^3$, $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. Evaluate or otherwise answer the following. (Answers without supporting work will not be considered.)

(a) Find $\|\mathbf{u}\|$ $= \sqrt{9+1+1} = \sqrt{11}$

(b) Evaluate $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v})$ $= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right) = 5 - 1 - 4 = 0$

(c) Are $\mathbf{u}$ and $\mathbf{v}$ parallel, perpendicular, or neither? $\vec{u} \cdot \vec{v} = 6 + 1 = 7 \neq 0$

$\vec{u} \neq k\vec{v}$ for any scalar k.

They are neither

(d) Are $\mathbf{v}$ and $\mathbf{w}$ orthogonal? $\vec{v} \cdot \vec{w} = 2 - 2 = 0$

They are orthogonal

(e) Find a unit vector in the direction of $\mathbf{w}$. $\|\vec{w}\|^2 = 1 + 1 + 4 = 6$

$$\frac{1}{\|\vec{w}\|} \vec{w} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{bmatrix}$$

(3) (20 points) Suppose A is a 5×5 matrix whose eigenvalues are found to be

$$\lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 = 1, \quad \lambda_4 = 2, \quad \text{and} \quad \lambda_5 = 3.$$

Use this information to answer the following questions.

(a) If $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda_1 = -1$, compute $A\mathbf{x}_1$.

$A\vec{x}_1 = -\vec{x}_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

(b) Evaluate $\det(A - 3I)$ $= 0$ since 3 is an eigenvalue

(c) If $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_4 = 2$, then $\|A\mathbf{x}_4\| =$

$\|A\vec{x}_4\| = \|2\vec{x}_4\| = |2|\,\|\vec{x}_4\| = 2\sqrt{1+9+1} = 2\sqrt{11}$

(d) Jack and Diane want to solve the equation $A\mathbf{x} = \mathbf{b}$ with this matrix and some given vector $\mathbf{b}$. Jack says they can solve the problem by computing A^{-1} because A has 5 different eigenvalues. Diane disagrees. She says that A doesn't have an inverse. One of them is correct. Who is correct, and how do you know?

Diane is correct. Since $\lambda_2 = 0$, $\det(A) = 0$

A is not invertible.

(4) (15 points) Let $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 16$$

$$= \lambda^2 - 4\lambda + 4 - 16$$

$$= \lambda^2 - 4\lambda - 12$$

(b) Determine the eigenvalues of A.

$$0 = \lambda^2 - 4\lambda - 12 = (\lambda-6)(\lambda+2)$$

The eigenvalues are $\lambda_1 = 6$, $\lambda_2 = -2$

(c) For each eigenvalue, find a basis vector spanning the associated eigenspace.

$$(A-6I) = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2\text{-free} \end{matrix} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$A+2I = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2\text{-free} \end{matrix} \quad \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

a basis is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

(5) (20 points) Let $\mathbb{P}_1$, the set of all polynomials of degree at most one, be given the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(1)\mathbf{q}(1).$$

For the vectors $\mathbf{p}(t) = 2 - t$, $\mathbf{q}(t) = 3 + 2t$, and $\mathbf{r}(t) = -2t$,

(a) Evaluate $\langle \mathbf{p}, \mathbf{q} \rangle$. $p(-1) = 3$, $p(1) = 1$ $q(-1) = 1$ $q(1) = 5$

$\langle \vec{p}, \vec{q} \rangle = 3(1) + 1(5) = 8$

(b) Find the distance between $\mathbf{q}$ and $\mathbf{r}$. $\vec{q} - \vec{r} = 3 + 4t$

$\|\vec{q} - \vec{r}\|^2 = (3-4)^2 + (3+4)^2 = 1 + 49 = 50$ $\|\vec{q} - \vec{r}\| = \sqrt{50}$

(c) Evaluate $\|\mathbf{p}\|^2$. $\langle \vec{p}, \vec{p} \rangle = (3)^2 + (1)^2 = 10$

(d) Find a unit vector in the direction of $\mathbf{p}$. $\vec{u} = \frac{1}{\|\vec{p}\|} \vec{p} = \frac{1}{\sqrt{10}} (2 - t)$

(e) Find a **nonzero** vector $\mathbf{s}(t) = s_0 + s_1 t$ in $\mathbb{P}_1$ that is orthogonal to $\mathbf{r}$. $\vec{r}(-1) = 2$ $\vec{r}(1) = -2$

$\langle \vec{s}, \vec{r} \rangle = (s_0 - s_1)(2) + (s_0 + s_1)(-2)$

$= 2s_0 - 2s_1 - 2s_0 - 2s_1 = -4s_1 = 0 \Rightarrow s_1 = 0$

so $\vec{s} = s_0$ an example would be $\vec{s}(t) = 1$

(6) (10 points) Find the least squares[1] best fit line to the data

x	-1	0	1	2
y	3	-6	3	-1

$mx + b = y$

$-1m + b = 3$

$0m + b = -6$

$1m + b = 3$

$2m + b = -1$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} m \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -6 \\ 3 \\ -1 \end{bmatrix}}_{\vec{b}}$$

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 & -2 \\ 2 & 4 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3/10 \\ 0 & 1 & -1/10 \end{bmatrix} \qquad m = \frac{-3}{10} \quad b = \frac{-1}{10}$$

$$\boxed{y = \frac{-3}{10}x - \frac{1}{10}}$$

[1] Recall that the normal equations are $A^T A\mathbf{x} = A^T\mathbf{b}$.