In this study, an attempt has been made to estimate the mean value of fecundability by various classical and non-classical methods. The classical methods are method of moments and method of maximum likelihood. The non-classical methods are standard Bayes, hierarchical Bayes, and empirical Bayes. For this purpose, the present study utilizes the Bangladesh Demographic and Health Survey (BDHS) data 1999-2000. In the analysis of fecundability, we have seen that the mean value of fecundability is about 0.046 whatever the estimation methods be.

ABSTRACT

The welfare of the people of a country is profoundly related to the size and composition of the population. The size and composition of the population are highly related to the fertility rate and growth rate of population of a country. The fertility rate and growth rate of a country are related and affected by fecundity and fecundability. Fecundity means the biological capacity of women to conceive and fecundability means the monthly probability of conception of those married women who did not use family planning before their first conception. Actually, this fecundability is called the mean natural fecundability. Fecundability has an opposite relation to the time interval required to conceive from the marriage to first conception. This time interval is known as the conception delay, conception interval or conception waits. Conception interval and fecundability are two important and interrelated fertility parameters and these are regarded as the most direct measures of fertility of a population. Thus the concept of fecundability is one of the principal determinants of fertility and one of the most important parameters for studying fertility patterns and human reproductive behavior in different societies.

Fecundability and marital fertility are linked through the following chain of variables: frequency of unprotected coitus, fecundability, exposure interval, birth interval, and marital fertility rate [1]. Fecundability affects fertility through its relationship with the average time required for a conception to occur and can also be considered as the transition probability for the passage from the susceptible state to pregnancy [2]. In a homogeneous population, fecundability is equal to the reciprocal of its mean conception delay [3] but for heterogeneous populations, the mean fecundabilities are usually modeled on two parameters [4, 5]. These parameters are estimated with the methods of moments and maximum likelihood.
It has been seen that a very few attempts have been made to estimate the mean fecundability for the Bangladeshi women as it is not a directly observable event. Thus, we know very little about such an important fertility parameter of Bangladeshi women. So, the fundamental purposes of the study are:

1. To estimate the mean fecundability for homogeneous women.
2. To estimate the mean fecundability for heterogeneous women by the methods of moments and maximum likelihood.
3. To estimate the mean fecundability for heterogeneous women by three Bayes methods.

DATA AND METHODOLOGY

This study utilizes the data from the 1999-2000 Bangladesh Demographic and Health Survey (BDHS). The survey was conducted during the period from November 1999 to March 2000. The survey was carried out under the authority of the National Institute of Population Research and Training (NIPORT) of the Ministry of Health and Family Welfare with funding from the US Agency for the International Development (USAID). A detailed description of the methodology of the data collection including the sample design for the survey can be found in [6].

A two stage probability sample design was used for the survey. At the first stage, a sample of areas was drawn and all the households in each of the selected areas were listed. At the second stage a nationally representative sample of 10268 households was selected, of which 9854 were successfully interviewed and finally out of 10885 married eligible women, a total of 10544 women of age less than 50 years belonging to the selected households were successfully interviewed. In this survey, only women who had ever been married were interviewed using the individual questionnaire. Among the 10544 ever-married females, 9696 were currently married and 848 formerly married. Among the currently married females 6828 were from rural areas and 2868 from urban areas. In order to estimate the fecundability for Bangladeshi women, we have extracted 1059 women out of 10544 women who have had at least one recognizable conception (regardless of outcome) before practicing contraception. We have excluded women who were pregnant before marriage. Since our study is based on birth history data, we exclude those conceptions of women occurring more than 5 years preceding the survey to avoid memory lapse of the respondents. Finally, we have also excluded those women who did not conceive during their first 15 years or 180 months of marriage, because women who fail to conceive within 15 years of their marriage are considered to be the primarily sterile. The outcome of a conception is (a) live birth, (b) non-live birth (induced abortion, spontaneous abortion, stillbirth and miscarriage) and (c) currently pregnant. Thus, we categorize the outcome of the first conceptions in the following way:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Live birth</td>
<td>1230</td>
</tr>
<tr>
<td>b) Non-live birth</td>
<td>304</td>
</tr>
<tr>
<td>c) Currently pregnant</td>
<td>217</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1751</strong></td>
</tr>
</tbody>
</table>
Excluding the cases of inadmissible conception intervals, we find 1681 women of which 1059 women have been found not using family planning before their first conception. The conception interval for women having any outcome was calculated by first close interval whereas it was calculated by open interval for currently pregnant women. Moreover, we have found 855 women who did not conceive during the survey time and hence the conception interval for these women is not possible to find [5, 6]. Since we have considered at least one conception whose outcome consists of live birth, non-live birth or currently pregnant, hence the conception interval can be computed in the following three ways:

1. Conception interval for live birth = (Date of first birth - Date of first marriage - 9)  
2. Conception interval for non-live birth = (Date of first pregnancy termination - gestation period - Date of first marriage)  
3. Conception interval for currently pregnant = (Date of currently pregnant - Duration of currently pregnant - Date of first marriage).

In this study, the analysis of BDHS (1999-2000) data yield us the following:

\[ n = \text{total number of women in our analysis} = 1059 \]
\[ x_i = \text{number of months required for conception of the ith women} \]
\[ y = \sum_{i=1}^{n} x_i = 22567 = \text{total number of conception months for the 1059 women}. \]
\[ \bar{y} = \text{mean conception wait} = \frac{22567}{1059} = 21.31. \]

RESULTS AND DISCUSSION

In the last centuries, demographers have employed a variety of techniques to study the mean value of fecundability and its distribution discussed in introduction. Among these techniques, the most commonly and widely applicable technique is to estimate the fecundability from the Geometric distribution directly when fecundability is considered to be constant. When fecundability varies among women, then it is typically estimated from Beta distribution of Type-I. To estimate mean fecundability from Beta distribution of Type-I, we need to estimate the two parameters of the distribution by some methods. One classical method is the method of moments and other one is the method of maximum likelihood. After estimating the parameters by each method, we will have two different Beta prior densities for the two methods of estimation, which will be used for Bayesian estimation of fecundability.

For analyzing the data on conception intervals, the Type-I Beta distribution is considered as a useful model. The model relies on the following assumptions:

1) Fecundability among couples is distributed as a Type-I Beta distribution. 
2) Conception is a random event, dependent on fecundability. 
3) The number of couples is adequately large. 
4) The fecundability \( \theta \) of each couple remains constant from month to month until recognizable conception.

The last assumption may be violated if the couples are temporarily separated, if the spouses intentionally change the timing of intercourse and if a miscarriage is not reported. During the period of separation the fecundability reduces to zero. However, if
the separation does not coincide with the fertile period during the month, then it will have no effect on fecundability [5].

Now if it is assumed that a woman’s fecundability had the constant value $\theta$ during the period of observation and $X$ is the random month of waiting to conception, then $X$ has the conditional geometric distribution with probability function given below:

$$P(X = x|\theta) = \theta (1 - \theta)^{x-1}; \quad 0 \leq \theta \leq 1; \quad x = 1, 2, 3, 4, 5, \ldots \ldots \ldots$$

$$E(x) = \frac{1}{\theta} \text{ and mean fecundability, } \theta = \frac{1}{E(x)}.$$

This is known as the conditional distribution of conception delay. Now, from the analysis of data, we have, $E(x) = \bar{y} = 21.31$ and hence the mean fecundability is 0.04692.

Now if $\theta$ varies among couples, then $\theta$ has the following density function:

$$f(\theta) = \frac{1}{\gamma(\alpha, \beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}; \quad \alpha, \beta > 0;$$

where $\alpha$ and $\beta$ are two unknown non-negative parameters which are to be estimated from the sample data to get prior densities for Bayesian analyses. Thus,

$$E(\theta) = \frac{\alpha}{\alpha + \beta}.$$

And the unconditional distribution of the conception delay $X$ is given by

$$P(X = x) = g(x) = \int_0^1 f(x, \theta) \, d\theta = \int_0^1 P(X = x|\theta)f(\theta) \, d\theta = \frac{\theta^{(\alpha+1)x + \beta-1}}{\gamma(\alpha, \beta)}$$

$$g(x; \alpha, \beta) = \begin{cases} 
\frac{\alpha}{\alpha + \beta}; & \text{if } x = 1 \\
\frac{\alpha \beta(\beta+1)(\beta+2)(\beta+3)\ldots\ldots(\beta+x-2)}{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)\ldots\ldots(\alpha+\beta+x-1)}; & \text{if } x \geq 2 
\end{cases}$$

a) Method of Moments:

Let $m_1$ and $m_2$ are the two observed first and second sample raw moments of the months required for women to conceive for the first time after marriage. The corresponding first two population moments of $X$ about origin, conditional on $\theta$

$$E(x) = \sum_{x=1}^{\infty} x \theta (1 - \theta)^{x-1} = \frac{1}{\theta}$$

and

$$E(x^2) = \sum_{x=1}^{\infty} x^2 \theta (1 - \theta)^{x-1} = \frac{2}{\theta^2} - \frac{1}{\theta}.$$

To obtain the unconditional moments of $X$, we have to substitute

$$E\left(\frac{1}{\theta^r}\right) = \int_0^1 \frac{1}{\theta^{(\alpha+\beta)r}} (1 - \theta)^{\beta-1} d\theta = \frac{(\alpha+\beta-1)(\alpha+\beta-2)\ldots\ldots(\alpha+\beta-r)}{(\alpha-1)(\alpha-2)(\alpha-3)\ldots\ldots(\alpha-r)}$$

to get

$$\mu_1' = \frac{\alpha + \beta - 1}{\alpha - 1} \text{ and } \mu_2' = \frac{2(\alpha + \beta - 1)(\alpha + \beta - 2)}{(\alpha-1)(\alpha-2)} - \frac{\alpha + \beta - 1}{\alpha - 1}.$$
Now equating $m_1$ with $\mu_1'$ and $m_2$ with $\mu_2'$, we get

$$\hat{a} = \frac{2s^2}{s^2 - m_1(m_1 - 1)} \quad \text{and} \quad \hat{\beta} = (\hat{a} - 1)(m_1 - 1)$$

where $s^2 = m_2 - m_1^2$ is the observed sample variance of the months required for the first conception of women after their marriage. It should be noted that $s^2 > m_1(m_1 - 1)$. Otherwise $\hat{a}$ will be less than two for which the theoretical second raw moment and variance are undefined and consequently no moment estimate is defined. Therefore, the method is restricted to the situations where the estimate of $\alpha$ is greater than 2. The analysis of the method of moments has shown that the moment estimators of $\hat{a}$ and $\hat{\beta}$ are moderately reliable only within a specified range of values of $\hat{a}$. Outside this range, either the estimates are extremely inefficient or their variances are not defined. Fortunately our moment estimates of $\alpha$ and $\beta$ for the 1059 women are 4.87 and 78.69 respectively. Thus, a prior density that will be used in three Bayes methods is:

$$f(\theta) = \frac{1}{\theta(4.87, 78.69)} \theta^{4.87-1} (1 - \theta)^{78.69-1}; \theta > 0; \quad a, b > 0;$$

and the mean fecundability is $E(\theta) = \frac{\hat{a}}{\hat{a} + \hat{\beta}} = \frac{4.78}{4.78 + 78.69} = 0.0572$

b) Method of Maximum Likelihood:

An alternative method of estimation, the maximum likelihood method, does not suffer the difficulties found in the method of moments. This method is not restricted to the particular range of the estimates and is defined for all permissible values of the parameters $\alpha$ and $\beta$. The maximum likelihood estimates can be derived in the following way: Let $k(x; \alpha, \beta)$ be the probability density function of a conception in month $x$ ($x=1,2,3,4,\ldots$) conditional on $\alpha$ and $\beta$. Then the likelihood function for the samples of $n$ values, given the parameters $\alpha$ and $\beta$ are defined as

$$L = \prod_{i=1}^{n} k(x_i; \alpha, \beta)$$

Now if $f(x)$ is the observed frequency of conception in month $x$ and $u$ is the highest recorded value of $x$ such that $\sum_{x=1}^{u} f(x) = n$; then we have

$$lnL = \sum_{x=1}^{u} f(x) \ln k(x_i; \alpha, \beta)$$

$$= \begin{cases} 
lna - ln(a + \beta); x = 1 \\
\sum_{x=1}^{u} f(x) [ln - f(1)]lna + \sum_{x=2}^{u} f(x) \sum_{j=0}^{x-2} ln(\beta + j) - \sum_{x=2}^{u} f(x) * \sum_{j=0}^{x-1} ln(\alpha + \beta + j); x \geq 2 
\end{cases}$$

$$= \begin{cases} 
 f(1)[lna - ln(a + \beta)]; x = 1 \\
lna + \sum_{j=0}^{x-2} ln(\beta + j) - \sum_{j=0}^{x-1} ln(\alpha + \beta + j); x \geq 2 
\end{cases}$$

The first order derivatives of $lnL$ with respect to $\alpha$ and $\beta$ are given by
Some Bayesian Analyses of Fecundability

\[ \frac{\delta \ln L}{\delta \alpha} = \sum_{x=1}^{n} f(x) \frac{\delta \ln L^{(x;\alpha,\beta)}}{\delta \alpha} \]
and
\[ \frac{\delta \ln L}{\delta \beta} = \sum_{x=1}^{n} f(x) \frac{\delta \ln L^{(x;\alpha,\beta)}}{\delta \beta} \]

The above equations are very complicated and it becomes cumbersome to maximize \( \ln L \) for the particular value of \( \alpha \) and \( \beta \) without the help of a computer. Consequently, we have used C++ language to find the estimates of \( \alpha \) and \( \beta \) for which \( \ln L \) is maximum. The procedure of finding the maximum value of \( \ln L \) consists of inserting initial values of \( \alpha_0 \) and \( \beta_0 \) as well as a small quantity (\( \epsilon \)) which is made progressively smaller as the iteration proceeds. C++ codes can be provided to interested readers upon request to the first author. The iteration is repeated until the largest value of \( \ln L \) for the particular value of \( \alpha \) and \( \beta \) is obtained. By using this procedure, the likelihood estimates of \( \alpha \) and \( \beta \) are found as 3.12 and 44.11. Thus another prior density that will be used in Bayesian analysis of fecundability is

\[ f(\theta) = \frac{1}{\beta(3.12, 44.11)} \theta^{3.12-1}(1-\theta)^{44.11-1}; \ a, b > 0; \]
and the mean fecundability is \( E(\theta) = \frac{\bar{a}}{\bar{a} + \bar{b}} = \frac{3.12}{3.12 + 44.11} = 0.0660 \)

These prior densities will be used to estimate the mean value of fecundability. While the moment estimates and maximum likelihood estimates related to our fecundability model discussed by [9]. We believe that the Bayesian analyses to follow are novel to this model.

i. Standard Bayes Method:

Our probability model of conception is:

\[ f(x|\theta) = \theta(1-\theta)^{x-1}; x = 1, 2, 3, 4, \ldots \ldots \]
and the joint density of the sample observations \( x_1, x_2, \ldots \ldots, x_n \) is:

\[ f(X|\theta) = \theta^n(1-\theta)^{\sum x - n}; \text{ where } X = (x_1, x_2, \ldots \ldots, x_n) \]

a) Considering the method of moments results, a reasonable prior density of \( \theta \) is

\[ h(\theta) = \frac{1}{\beta(4.87, 78.69)} \theta^{4.87-1}(1-\theta)^{78.69-1}; \]

The posterior density of \( \theta \) is:

\[ g(\theta|X) = \frac{h(\theta) f(x|\theta)}{\sum h(\theta) f(x|\theta)} = \frac{1}{\beta(4.87, 78.69) \theta^{n+4.87-1}(1-\theta)^{\sum x + 78.69 - n - 1}} \]

So for the squared error loss, the Bayes estimator is the posterior mean and the mean fecundability is

\[ \hat{\theta} = \frac{n + 4.87}{\sum x + 4.87 + 78.69} = \frac{1059 + 4.87}{22567 + 4.87 + 78.69} = 0.04696. \]
b) Considering the maximum likelihood results, another reasonable prior density of \( \theta \) is
\[
h(\theta) = \frac{1}{h(3.12, 44.11)} \theta^{3.12-1} (1 - \theta)^{44.11-1}.
\]
The posterior density of \( \theta \) is
\[
g(\theta | X) = \frac{1}{\theta(n + 3.12, \Sigma x - n + 44.11)} \theta^{n+3.12-1} (1 - \theta)^{\Sigma x - n + 44.11-1}
\]
So the Bayes estimator is the posterior mean and the mean fecundability is
\[
\hat{\theta} = \frac{n + 3.12}{\Sigma x + 3.12 + 44.11} = \frac{1059 + 3.12}{22567 + 3.12 + 44.11} = 0.04696
\]

ii. Hierarchical Bayes:
Recall, our probability model of conception is:
\[
f(x | \theta) = \theta (1 - \theta)^{x-1}; x = 1, 2, 3, 4, \ldots \ldots \ldots
\]
Here, \( y = \sum_{i=1}^{n} x_i \) has a negative binomial distribution and is also a sufficient statistic. Therefore, we can write:
\[
g(y | \theta) = \binom{y-1}{n-1} \theta^n (1 - \theta)^{y-n}
\]
If we let prior density of \( \theta \) depend on another unknown parameter \( b \) given by
\[
h(\theta | 1, b) = \frac{1}{h(1, b)} \theta^{1-1} (1 - \theta)^{b-1}; 0 < \theta < 1
\]
Now let the pdf of the random variable \( b \) be given by
\[
k(b) = \frac{1}{\lambda} e^{-\frac{1}{\lambda} b}; b > 0
\]

Now,
\[
h_1(\theta | y, b) = \frac{h(\theta, y, b)}{h(y, b)} = \frac{h(\theta, y, b)}{h(y, b) \theta b} \sim B(n + 1, y + b - n) \ldots \ldots \ldots (a)
\]
and
\[
h_1(b | y, \theta) = \frac{h(\theta, y, b)}{h(y, \theta)} = \frac{h(\theta, y, b)}{h(y, \theta) \theta b} \sim \Gamma\left(2 \frac{b}{1-\lambda \ln(1-\theta)} \right) \ldots \ldots \ldots (b)
\]
where
\[
h(\theta, y, b) = g(y | \theta) * h(\theta | 1, b) * k(b) = \binom{y-1}{n-1} \theta^n (1 - \theta)^{y+b-n-1} \frac{1}{\lambda} e^{-\frac{1}{\lambda} b}
\]
\[
h(y, b) = \int_0^1 h(\theta, y, b) d\theta = \frac{1}{\binom{y-1}{n-1}} \theta^n (1 - \theta)^{y+b-n-1} \frac{1}{\lambda} e^{-\frac{1}{\lambda} b} d\theta
\]
\[
= \binom{y-1}{n-1} \frac{1}{\lambda} e^{-\frac{1}{\lambda} b} \int_0^1 \theta^n (1 - \theta)^{y+b-n-1} = \binom{y-1}{n-1} \frac{b}{\lambda} e^{-\frac{1}{\lambda} b} \ln(1-\theta)
\]
and
\[
h(y, \theta) = \int_0^\infty h(\theta, y, b) db = \int_0^\infty \binom{y-1}{n-1} \theta^n (1 - \theta)^{y+b-n-1} \frac{1}{\lambda} e^{-\frac{1}{\lambda} b} db
\]
\[
= \frac{\binom{y-1}{n-1} \theta^n}{\lambda} \int_0^\infty b^{2-1} e^{-\frac{b}{\lambda} \ln(1-\theta))} db
\]
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\[ e^{(y-n-1)\ln(\theta)} \int_0^{\infty} b^{2-1} e^{-b \frac{(1-\ln(1-\theta))}{\lambda}} db \]

\[ = \frac{(y-n-1)\theta^n e^{(y-n-1)\ln(\theta)}}{\lambda} + \frac{1}{\left(\frac{1}{\lambda} - \ln(\theta)\right)^2} \]

Now using the Gibbs Sampler technique on equations (a) and (b) with the help of Mathematica, we get the mean fecundability from the initial values of \( b = 5 \) and \( \lambda = 2 \) as 0.04694. The Gibbs Sampler technique was used with 1500 replications to estimate the above value of \( \theta \). The Mathematica code is provided in Appendix-1.

### iii. Empirical Bayes:

The empirical Bayes model consists of the first two lines of the Hierarchical Bayes and instead of attempting to model the parameter with a pdf as in Hierarchical Bayes, Empirical Bayes methodology estimates the parameter directly from the data [7].

Again, our probability model of conception is:

\[ f(x_1) = \theta(1 - \theta)^{x-1}; x = 1, 2, 3, 4, \ldots \ldots \]

Let the prior density of \( \theta \) be:

\[ h(\theta | b) = \beta(1, b) = \frac{1}{\beta(1, b)} \theta^{1-1} (1 - \theta)^{b-1} \]

Now we have estimated \( b \) in the following way:

\[ g(X, \theta | b) = \frac{h(X, \theta | b)}{h(\theta)} = \frac{f(X | \theta) h(\theta | b) h(\theta)}{h(\theta)} = f(X | \theta) h(\theta | b) = b \theta^n (1 - \theta)^{2n-i+b-n-1} \]

where \( X = (x_1, x_2, \ldots, x_n) \).

Let us take \( y = \sum_{i=1}^n x_i \) for the convenience of estimating \( b \). Then the likelihood function will be as follows:

\[ L = \int_0^1 g(X, \theta | b) d\theta = \int_0^1 b \theta^n (1 - \theta)^{2n-i+b-n-1} d\theta = b \beta(n+1, y+b-n) \]

\[ \ln L = \ln b + \ln (n!) - \sum_{i=0}^n \ln \Gamma(y+i) \]

\[ \frac{\delta \ln L}{\delta b} = \frac{1}{b} - \sum_{i=0}^{n+b-1} \frac{1}{y+b-1} = 0. \]

By plotting the above in Mathematica, we can roughly estimate the solution of \( \frac{\delta \ln L}{\delta b} = 0 \) as \( b = 20 \). Using this initial value in a numerical routine in Mathematica yields the root \( b = 20.7895 \). For details, see Appendix-2. Now the Bayes estimate of fecundability by the Empirical method is the mean of posterior pdf for squared error loss. Now the Posterior pdf is

\[ k(\theta | X, b) = \frac{1}{b(n+1) \sum_{i=1}^n x_i+b-n-1} \theta^{n+1-1} (1 - \theta)^{2n-i+b-n-1} \]
and the empirical Bayes estimate is
\[
\hat{\theta} = \frac{n+1}{\sum_{i=1}^{n} x_i + b + 1} = \frac{1060}{22567.29 + 20.7859 + 1} = 0.04692.
\]

CONCLUSION AND FUTURE STUDY

From the above discussion of estimating mean fecundability by various classical and non-classical methods, we can say the mean fecundability among Bangladeshi women is 0.046 whatever be the methods of estimation. One reason for this stable value of mean fecundability is attributed to the large value of sufficient statistics i.e. 
\[ y = \sum_{i=1}^{n} x_i = 22567. \] We also note that the Bayesian analyses seems to fit this conception model quite well and are eager to see it applied in further studies. For future study in this subject matter, we suggest the reader to find minimax estimator, which minimizes the maximum risk or finding a Bayes estimator with constant risk. To get into the point of minimax estimator, one might end up with the upper limit of constant risk by applying Jensen’s inequality to convex loss function.

REFERENCES

APPENDIX-1

<<Statistics 'Continuous Distributions'
X=Table[Random[Gamma Distribution[3,5]],{i,1,10}]
Y=Table[Random[Beta Distribution[1060,21513.29]],{i,1,1}]
{0.0475708}
Z=Table[Random[Gamma Distribution[2,0.5489]],{i,1,1}]
{1.00259}
b=5
n=1500
sum=0
Do[{a=Random[BetaDistribution[1060,21508.29+b]],b=Random[GammaDistribution[2,
2/(1-2Log[1-a])],sum=sum+a,Print[a]Print[b]},{i,1,n}]
sum
avgtetasum/n

APPENDIX-2

Clear[b,LL]
LL[b_]:=1/b-Sum[1/(22567.29+b-i),{i,0,1059}]
Plot[LL[b],[b,0,90]]

2.5
2
1.5
1
0.5

Clear[b]
FindRoot[LL[b]==0,{b,50}]
{b->20.7859}
bhat=20.78583471403103
20.7859
Thetahat=1060/(22567.29+bhat+1)
0.0469253