Projective Geometry for All

Meighan Dillon

Meighan Dillon (mdillon@spsu.edu) received a Ph.D. in mathematics from the University of Virginia and a B.A. in mathematics from Hunter College, where she began her love affair with geometry. She has taught for most of her career at Southern Polytechnic State University, soon to become part of Kennesaw State University, in metropolitan Atlanta, and in her spare time enjoys yoga, swimming, playing the piano, and endeavoring gamely to learn French.

Some of us never got over the thrill of learning projective geometry for the first time, and why would we?

Definition. A *projective plane* is a set of points and lines that satisfy the following.

A1. Two points determine a unique line.

A2. Two lines determine a unique point, the point of intersection.

A3. There are four points, no three of which are collinear.

Defined by three spare, perfectly balanced axioms, the projective plane is a jewel-like model of symmetry. Its elegance lulls us into a small rapture that shatters like glass into unsettling consequences. What a great topic for wooing anyone who thinks mathematics is dry!

As a natural extension of Euclidean geometry, projective geometry may—or may not—be invited for an appearance in the geometry courses we teach math education students and other interested parties. Aside from its importance in modern academic geometry, projective geometry's connections in history and culture should put it on the A list of celebrity topics we want all our important people to meet. When we ran across projective geometry for the first time, for example, most of us were told that, in the form of perspective, projective geometry had its beginnings in Renaissance art, and that it then became, not just interesting, but central to academic mathematics 450 years later—a fantastic reversal of the usual trajectory. A cogent, directed answer to the question of how projective geometry made its strange voyage from Renaissance art to academic mathematics—and why it took so long—is not easy to come by. Indeed, the practice of perspective, the mathematical theory of perspective, academic projective geometry, and the history and transmission of all of the above are distinct threads and delicately intertwined. Disentangling them is not for the faint of heart! But experience working with the simplest idea of perspective in art can prepare even skittish students for a modest helping of projective geometry and, with it, a slice of history worthy of any conversation involving cross-currents between society and culture and the world of mathematics and science.

Starting with the use of perspective in Renaissance art, our narrative sketches a program that can be used to introduce students at all levels to both the artistic background

http://dx.doi.org/10.4169/college.math.j.45.3.169
MSC: 00A66, 51A05, 97G99

and the mathematical basis of projective geometry. Each section has exercises that lead students to engage more directly with the mathematics through calculations and concept discovery. In introducing students to projective geometry using historical narrative and motivation, we may open their eyes and minds to the fact that mathematics, like all human endeavors, develops within a cultural and social context.

The story we trace follows a circuitous route and elements of it may not be familiar to all audiences. As such, readers may benefit from consulting sources cited at the beginning of each section.

Perspective in art

Here we describe how a problem in art was addressed during the Renaissance using the geometry and optics of Euclid. The reader may find it useful to have [**3**], [**10**] or [**12**], and [**15**] at the ready.

The convincing representation of a three-dimensional subject on a flat surface is a problem that artists have wrestled through the ages. There is abundant evidence, going back to antiquity, that the ability to create pictures that tricked the eye or gave a strong sense of space has long been highly prized [**1, 13, 15**]. What changed in the Renaissance was that artists became "obsessed by mathematics" [**13**] and sought rules of perspective based on mathematical principles. The oldest surviving written rules for perspective (by which we always mean one-point linear perspective)—one technique among many for rendering three-dimensional scenes on a picture plane—date to 1435 with the appearance of Leon Battista Alberti's *De Pictura*.

How does one address the problem of representing a three-dimensional scene on a flat surface? Alberti's approach is to marry Euclid's model of vision to his geometry. Each of the following assumptions is expressly articulated in Euclid's *Optics*, either as a definition, a theorem, or within the argument for a theorem [**3**].

1. Identify an observer's eye with a point.
2. Assume vision occurs along a line between the eye and each visible point on an object.
3. Parallel lines seen from a distance appear to grow closer together.
4. Objects seen in the same angle are perceived as having the same size.
5. The rays between a viewer's eye and a perceived object form a cone with the eye at the apex, the object at the base.

Exercise 1. What simplifications are implicit in the five assumptions stated above from Euclid's *Optics*? Which assumptions may be wrong and what are some possible alternatives?

An observer whose eye is at a point O stands in front of a picture plane π. (The notation and terminology here are all borrowed from [**1**] and [**7**].) There is a real-world scene behind π and the problem is to depict the scene on π just as it appears to the observer. The point O and a visible point A in the real world determine a line that intersects π at A'. The artist should depict the real world point A at A' on the canvas.

Exercise 2. Sketch the set-up described in the previous paragraph.

Albrecht Dürer (1471–1528) was an accomplished painter, engraver, and printer. Students may recognize Dürer's artwork (his study of hands in prayer is ubiquitous) but they may be surprised to learn that not only was Dürer was a serious mathematician,

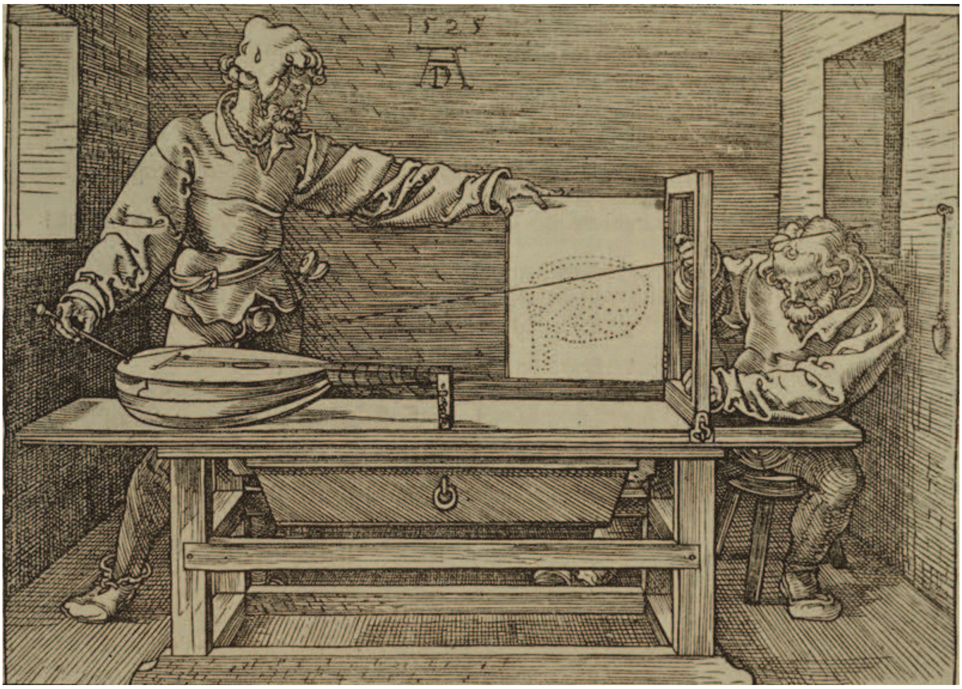

Figure 1. Albrecht Dürer, *Man Drawing a Lute*, 1525. In the collection of the Metropolitan Museum of Art, New York. Image from Wikimedia.

his art had mathematical content. Dürer was fascinated by perspective and went to Italy to study it [**15**]. He illustrated Alberti's methods in several pictures.

Exercise 3. What is each character doing in *Man Drawing a Lute* (Figure 1)? Where are the points O, A, and A'? How will the artist in the picture accomplish his perspective drawing?

Alberti's treatise gave a recipe for making a picture in perspective, but not an explanation for why the recipe worked. Using Propositions 2 and 3 in Book XI of the *Elements* [**12**] and the definition of conic sections as curves at the intersection of a cone and a plane, the discovery of which is usually attributed to Menaechmus, who was about a generation older than Euclid [**15**], students can themselves justify the rules of perspective.

Exercise 4. Show that when an artist uses Alberti's method,

(a) straight lines in the real world should be rendered as straight lines on π,
(b) real-world parallels extending away from π should be depicted as convergent,
(c) conic sections—ellipses, parabolas, and hyperbolas—should be rendered as conic sections, possibly of different types.

Perspective was of particular interest to 15th-century architects who wanted to revive the classical building style. Brunelleschi, renowned architect and signal influence on Alberti, was likely "the first to paint genuine perspective compositions" [**1**]. Brunelleschi famously painted two architectural scenes, both lost, to show off his technique. One was of the Baptistery in Florence, an octagonal structure dating to the 11th century. The Baptistery picture was meant to be reflected in a mirror and viewed through a pinhole [**1, 6**]. The region in the picture above the building was silvered, presumably to reflect the sky, enhancing the viewer's sense of seeing the real world scene itself [**6**].

Exercise 5. Why would Brunelleschi have insisted on the viewer observing his painting through a pinhole? Why use a mirror?

Going back to the picture plane π and the real-world scene to be depicted, let γ denote the ground upon which the observer stands. The line where γ and the plane of π intersect is g, the *ground line* (see Figure 2). The orthogonal projection of the eye point O upon π is the *principal vanishing point*, P. The line in π through P and parallel to g is the *horizon*, h. *Orthogonals* represent a collection of real-world parallel lines extending into the distance away from g. They are depicted on π as converging at P. *Transversals* are lines running through the orthogonals, parallel to g.

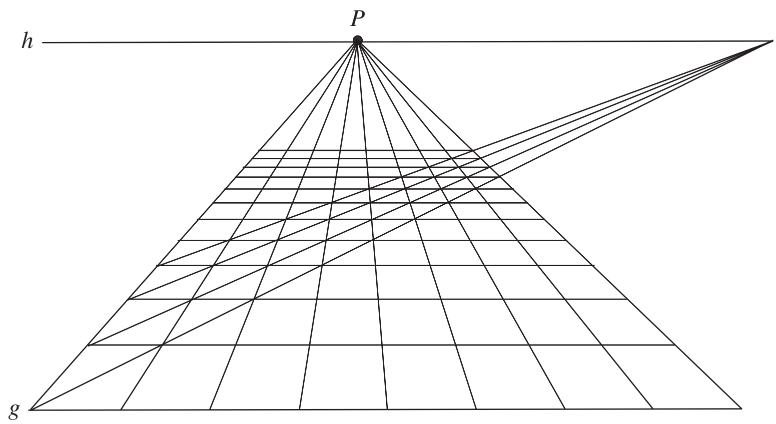

Figure 2. A square tiled floor in perspective.

Alberti gave instructions for rendering a square tiled floor, identified with γ. The artist places P somewhere in the picture plane. The lower edge of the canvas, g, is divided into equally spaced subintervals, as in Figure 2. The artist constructs orthogonals, to depict the right and left sides of the square tiles, by connecting P to each point defining the partition of g. The trick is spacing the transversals correctly. Alberti's method for arranging the transversals exactly where they appear employs a detailed side view of the construction with careful placement of γ, π, g, etc. [**1**]. He counsels using diagonals to check whether the perspective is right, but you can use diagonals to find correct placement of the transversals to begin with, which is what we did in Figure 2: The line connecting the lower left point of g, say, and a point at the upper right on h—on or off π—is a diagonal for squares in every row. Intersection points with orthogonals tell you how to place the transversals (see [**17**] for an online video tutorial). Once you know this diagonal trick, you can experiment with renderings, which will not only dazzle your friends but renew your respect for the artists and craftsmen who carried out this process using hand tools and careful measurements on the wet plaster of ceilings and walls. (See [**6**] for a plethora of stories, including detailed analyses of frescoes.)

Exercise 6. Make several sketches of a square tiled floor, by hand or using vector-based graphics software, experimenting with different vanishing points, orthogonals, and diagonals. Note that the square tiles establish scale at different points in the picture plane and assume that the squares are each one foot by one foot. Using the grid and the best of your drawing ability, put some figures into your pictures.

Exercise 7. In your perspective drawing of a tiled floor, the transversals get closer together as they get closer to the horizon line. Where does Euclid refer to this phenomenon in the *Optics*?

Figure 3. Andrea Mantegna, *Lamentation over the Dead Christ*, also known as *Foreshortened Christ*, c.1490. In the collection of the Pinacoteca di Brera, Milan Italy. Image from Wikimedia.

Exercise 8. Can you use transversals to explain the effect that Andrea Mantegna achieved in *Lamentation over the Dead Christ* (Figure 3)?

Mantegna's spectacular illustration of foreshortening dates to around 1490, by which time expertise in perspective had become de rigueur for Renaissance artists. Art like this can get students thinking about scale, and about how our brains interpret a picture to get a perception of scale, so critical in the real world, yet meaningless in projective world... which brings us to projective world.

Projective geometry

Here we start with Pappus' Theorem, not to prove, but to consider in terms of perspective. We then guide the reader to develop a few models of the real projective plane. The reader may again want Book XI of the *Elements* nearby while reading this section.

Mapping real-world points into the picture plane π using the eye point O is *central projection* through O. In Coxeter's words, "plane projective geometry may be described as the study of geometrical properties that are unchanged by 'central projection'" [**5**].

The next exercise refers to Pappus' Theorem, illustrated in Figure 4. Here, PQ designates the line determined by points P and Q, and $PQ \cap P'Q'$ is the point of intersection of lines PQ and $P'Q'$.

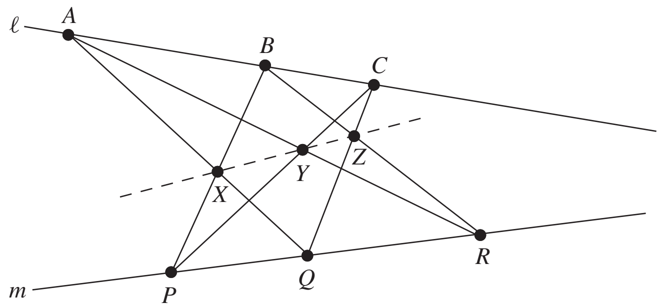

Figure 4. The Pappus configuration.

Exercise 9. Suppose A, B, C are points on a line ℓ, and P, Q, R are points on a second line m. Let $X = AQ \cap BP$, $Y = AR \cap CP$, and $Z = BR \cap CQ$. In particular, assume that all lines in question intersect. Pappus of Alexandria, the fourth-century geometer, used the tools of Euclid's *Elements* to prove that X, Y, Z must be collinear [**5**]. Argue that Pappus' Theorem, typically cited as the oldest theorem in projective geometry, concerns a configuration unchanged by central projection.

Having lived with perspective for a little while, students may be able to ease into a model of $\mathbb{R}P^2$, the real projective plane. While the idea of a line standing for a point is itself hard to swallow, the perspective problem gives it context and a real-world point A. Our problem is to associate to A an object P in the real world (or its idealization, $\mathbb{R}^3$) that will play the role of "point" in $\mathbb{R}P^2$. Our only criteria for P are that it must have a natural association to A and it must remain unchanged by central projection of A through O. The point A' depends only on the position of π, but we allow π to occupy any position in space. Since π can be placed to intersect the line OA at any point, the line OA must itself be P. With the help of the *Elements* again, students can come around to the idea that, in this model, planes through O stand for projective lines.

Each line through the origin in $\mathbb{R}^3$ intersects the unit sphere twice. This is a second model for $\mathbb{R}P^2$, the sphere with antipodal points identified. It brings new insights into the real projective plane.

Exercise 10. If a point on the sphere model of $\mathbb{R}P^2$ is represented by the intersection of the unit sphere and a line through the origin in $\mathbb{R}^3$, how should we represent a line on the sphere model of $\mathbb{R}P^2$?

Exercise 11. Suppose you stand at a point on the sphere model of $\mathbb{R}P^2$ and your sight lines follow projective lines. If you could see forever, what would you be looking at?

The axiomatic definition brings out other properties of projective planes.

Exercise 12. Show that the lines-through-the-origin model of $\mathbb{R}P^2$ satisfies the axioms A1–A3. Show that the sphere model does as well.

The most naïve view of $\mathbb{R}P^2$ is a Euclidean plane that does not have parallels and enjoys no acknowledgment of length or angle measure. A proper model of $\mathbb{R}P^2$ based on this idea requires appending to the Euclidean plane a point of intersection for each family of parallel lines, the point in $\mathbb{R}P^2$ where Euclidean parallels intersect. Inexperienced students can do the following exercise even if they have not worked through that construction.

Exercise 13. In Euclidean geometry, we consider quadrilaterals, arbitrary four-sided plane figures. The projective analogue to a quadrilateral is a *quadrangle*. A (complete) quadrangle is determined by four *vertices*, that is, four coplanar points, no three of which are collinear. Each pair of vertices determines a *side* of the quadrangle. Two sides of a quadrangle that do not intersect in a vertex are *opposite*. Since any two lines in a projective plane intersect, so do opposite sides of a quadrangle. We call the point of intersection of opposite sides a *diagonal point*. How many sides does a quadrangle have? Sketch a few quadrangles using Euclidean points and lines, taking care to arrange the points so that any two lines intersect. Identify opposite sides and diagonal points.

The minimal configuration of points and lines that satisfies A1–A3 is called the Fano plane. Duality is the principle that, in any projective plane, points and lines have symmetric roles. With axioms A1–A3, students can access both the Fano plane and duality.

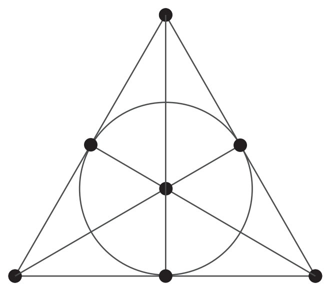

Figure 5. The Fano plane.

Exercise 14. Prove that the configuration in Figure 5, a Fano plane, satisfies the axioms for a projective plane. What are the points and what are the lines? Why does it not work to have a smaller configuration satisfying A1–A3? Switch the roles of lines and points and verify the axioms again this way.

Exercise 15. State the dual of axiom A3 and show that the dual property holds in a projective plane.

Exercise 16. Show that if we take planes through the origin in $\mathbb{R}^3$ as points and lines through the origin as lines, then we have another model for $\mathbb{R}P^2$.

Exercise 17. A triangle is a set of three noncollinear points and the lines they determine. What is the dual of a triangle? Where are the triangles in the Fano plane?

Exercise 18. What is the dual of a quadrangle? Find the quadrangles in the Fano plane. Identify opposite sides and diagonal points for the quadrangles you found. Comparing these quadrangles to the ones you sketched in $\mathbb{R}P^2$, what peculiarities do you notice about diagonal points?

What next?

This section briefly treats Desargues' theorem and the history of the parallel postulate. The reader may appreciate ready access to [**9**, ch. 20], which treats general projective planes and deals with the associated algebra. Anyone rusty on the history of the parallel postulate will benefit from having [**8**] at hand.

Pappus' Theorem was the first in projective geometry and dates to the fourth century. Desargues' Theorem is typically cited as the second and dates to the mid-17th century.

Consider triangles ABC and $A'B'C'$. If $A'B'C'$ is a central projection of ABC, with O the center, then the triangles are *perspective from the point* O and we say O is a *point of perspectivity* for the two triangles. Now assume that corresponding lines of the triangles intersect. Let ℓ be the line determined by points $AB \cap A'B'$ and $AC \cap A'C'$. If $BC \cap B'C'$ also lies on ℓ, then the triangles are *perspective from the line* ℓ, which is then the *axis of perspectivity* for the two triangles. Simply put, Desargues' Theorem says triangles are perspective from a point if and only if they are perspective from a line (see Figure 6).

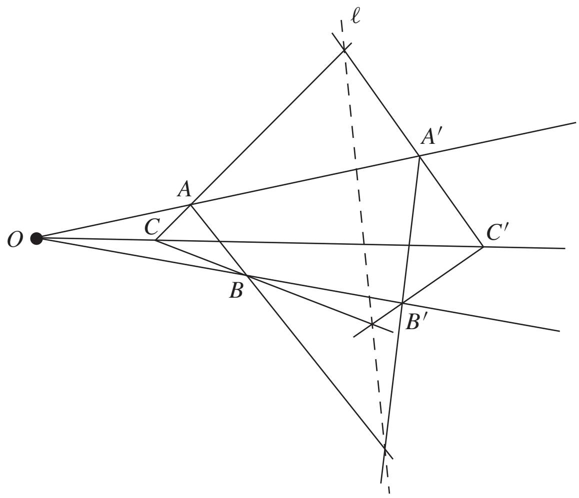

Figure 6. The Desargues configuration.

Exercise 19. Objects perspective from a point are related by central projection. We understand this visually by thinking of the point of perspectivity as a point where we can visually line up corresponding points of two objects so they appear to overlap perfectly. Objects perspective from a line enjoy the dual property. How can we understand that visually?

Exercise 20. Can you find triangles in the Pappus configuration (Figure 4) that are perspective from a point? What are the point and axis of perspectivity?

Exercise 21. Find triangles in the Fano plane that are perspective from a point. What is their axis of perspectivity?

The geometric interpretation of Desargues' Theorem is that it holds if and only if the projective plane in question can be embedded in a three-dimensional projective space. The question as to whether this is possible is connected to questions about whether a given projective plane can be modeled using a module over a ring or a vector space over a field. A Desarguesian plane can be modeled using what is essentially a three-dimensional vector space, which can be a subspace of a four-dimensional vector space that models a three-dimensional projective space. General projective planes, the ones defined by our three axioms, can be strange kettles of fish, unable to survive outside their own two-dimensional world. The world of finite projective planes, like the Fano plane, is itself so mysterious that the simple question of how many points and lines it is possible to have in such a thing remains open. These types of objects were a long way from being recognized in the 17th century, though. Indeed, the easy case of Desargues' theorem is the one in which the triangles occupy two different planes, the case most closely related to the real-world perspective problem. Desargues captured the connection between two and three dimensions quite consciously in his theorem, (see the notes in [**14**]), which properly belongs to projective geometry rather than the theory of perspective.

Desargues' work on projective geometry, while of enduring value, slipped almost directly from Desargues' hands into obscurity, where it remained for 200 years [**15**]. It was not until well after the appearance of Jean-Victor Poncelet's work, the first comprehensive treatise on projective geometry, which was finally published in toto in 1868, that projective geometry was fully recognized in the academy. It was perhaps Felix Klein's work that gave projective geometry that final push, but well may we ask, what took so long?

Western Europe from the Renaissance through the 18th century was fertile ground for mathematics, but during those 500 years, academic geometry labored beneath a cloud that grew darker and more menacing with each succeeding generation: the controversy surrounding Euclid's Fifth Postulate. It is exquisite irony that the resolution of the controversy meant at once that Euclid was correct in taking the existence of unique parallels as an assumption, not a theorem, and at the same time that space was not necessarily Euclidean. Lobachevsky and Bolyai initiated the resolution of the controversy with their discoveries of the consistency of non-Euclidean geometries in the early 19th century. The controversy was entirely resolved by the end of the century with the work of Beltrami, Poincaré, and Klein, each of whom developed models of a hyperbolic plane constructed inside Euclidean space. It was in 1854, though, that the true weight of the discovery of non-Euclidean geometries was brought fully to bear on the collective imagination of the mathematics community in Europe. In that year, Riemann delivered his post-doctoral (*Habilitation*) lecture on the foundations of geometry. He may as well have delivered a thunderbolt. In his lecture, Riemann tendered the concept of an n-manifold, a space of arbitrary dimension, a space that itself may be curved [**14**]. The work astounded everyone—most importantly, Gauss—and it changed everything, deeply, for good. In terms of subject matter, Riemann's work had nothing to do with perspective or with projective geometry, but there was a critical cultural connection.

Riemann's work freed people to think of space and the foundations of geometry in a new way. Sophus Lie's work on transformation groups, Einstein's theory of relativity, and Klein's Erlanger Programm can all be traced to Riemann's influence. Klein, one of the people who worked out the coordinatization of projective geometry, also understood its applications to non-Euclidean geometries [**5**]. Thus, by late in the 19th century, led by the hand of Felix Klein, a giant of mathematics and its pedagogy, projective geometry had become, to paraphrase Arthur Cayley, all geometry [**5**].

Conclusion

This narrative is a simplification of a story with many players and byzantine complexity. The hope is that it might serve as a springboard for further study of geometry by students with general backgrounds. An object of simplicity, beauty, and ubiquitous utility, the projective plane begs to be featured in courses for general audiences. It amply rewards students and instructors with more questions than answers.

The list of references at the end of the paper includes sources for mathematics courses for liberal arts students or art courses for students willing to take on some mathematical ideas, among them [**7, 11, 13**]. A source for the intrepid that assumes limited mathematical experience in a treatment of the nature of space and topology is [**16**]. A lively and expert source on Renaissance art and mathematics is [**6**]. Covering the history of perspective in great detail is [**1**]. Anyone interested in the history of the controversy surrounding the parallel postulate will be rewarded by reading [**8**]. For a more advanced and algebraic take on projective geometry, see [**2**]. Sources on mathematics history are abundant and well-known, but [**15**] is particularly good for geometry. Perhaps less well-known but amply edifying is [**4**]. Anyone interested in the history of mathematics will enjoy the Heath edition of the *Elements* of Euclid [**10**]. David Joyce's website treatment of the *Elements* is a wonderful source that students should be encouraged to explore [**12**]. Anyone with more than a passing interest in projective geometry should study [**5**].

Acknowledgment. The author thanks the referees and the editor for their careful reading and thoughtful suggestions.

Summary. This article treats projective geometry as arising from the perspective problem addressed by Renaissance artists while sketching a program for teaching the material to students at all levels.

References

1. K. Andersen, *The Geometry of an Art: The History of the Mathematical Theory of Perspective from Alberti to Monge*, Springer, New York, 2007.
2. J. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002) 145–205.
3. H. E. Burton, The optics of Euclid, *Journal of the Optical Society of America* **35** (1945) 357–372.
4. J. L. Coolidge, *A History of Geometrical Methods*, Dover, Mineola NY, 2003.
5. H. S. M. Coxeter, *Projective Geometry*, Blaisdell, New York, 1964.
6. J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford Univ. Press, Oxford, 1997.
7. M. Frantz and A. Crannell, *Viewpoints: Mathematical Perspective and Fractal Geometry in Art*, Princeton Univ. Press, Princeton NJ, 2011.
8. J. V. Grabiner, Why did Lagrange "prove" the parallel postulate? *Amer. Math. Monthly* **116** (2009) 3–18, http://www.maa.org/programs/maa-awards/writing-awards/why-did-lagrange-prove-the-parallel-postulate
9. M. Hall, Jr., *The Theory of Groups*, Chelsea, New York, 1976.
10. T. L. Heath, *The Thirteen Books of Euclid's Elements*, 2nd ed., Dover, Mineola NY, 1956.
11. M. Holt, *Mathematics in Art*, Studio Vista, London/Van Nostrand Reinhold, New York, 1971.
12. D. E. Joyce, Euclid's Elements, 1998, http://aleph0.clarku.edu/~djoyce/java/elements/elements.html.
13. A. Pipes, *Foundations of Art and Design*, 2nd ed., Laurence King, London, 2008.
14. D. E. Smith, *A Source Book in Mathematics, Vol. 2*, Dover, New York, 1959.
15. J. Stillwell, *Mathematics and Its History*, 3rd ed., Springer, New York, 2010.
16. J. R. Weeks, *The Shape of Space*, 2nd ed., Marcel Dekker, New York, 2002.
17. N. J. Wildberger, WT32: Projective geometry and perspective, *WildTrig*, 2008, http://www.youtube.com/watch?v=qJGmX83MNUY.