Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 10: Path independence and conservative vector fields

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1 Path independence

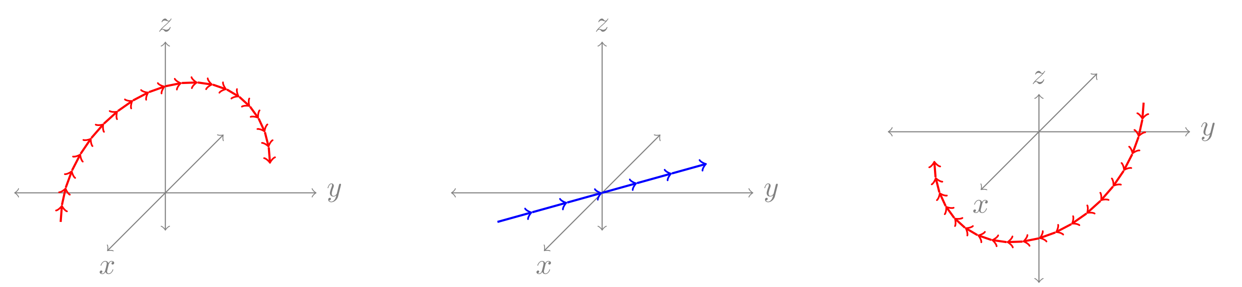
We return to the (flow) vector line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C M \,\mathrm{d}x + N \,\mathrm{d}y + P \,\mathrm{d}z.$$

We've already seen in an example that *sometimes*, but not *always*, this integral does not depend on the exact path C, only on where C starts and ends. This property is called **path independence**, and the vector fields $\mathbf{F}$ for which this integral is path-independent are called **conservative fields**.

1.1 Another example of path independence

Here is another example. Let $\mathbf{F} = (y-1)\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}$. We will first compute the integral of $\mathbf{F}$ along the semicircle in Figure 1a, which it's a good exercise to try to parameterize.



(a) A path from $(1,-1,0)$ to $(-1,1,0)$ along a semicircle

(b) A path from $(1,-1,0)$ to $(-1,1,0)$ along a straight line

(c) A path from $(-1,1,0)$ back to $(1,-1,0)$ along a semicircle

Figure 1: Three different paths between $(-1,1,0)$ and $(1,-1,0)$

The path in Figure 1a can be parameterized as

$$\mathbf{r}(t) = (\cos t, -\cos t, \sqrt{2}\sin t), \qquad t \in [0, \pi].$$

Think of this as specifying a "horizontal radius vector" of $\mathbf{a} = (1,-1,0)$ and a "vertical radius vector" of $\mathbf{b} = (0,0,\sqrt{2})$. (Why $\sqrt{2}$? Because the points $(1,-1,0)$ and $(-1,1,0)$ are both at distance $\sqrt{2}$ from the center of the circle.) Then, $\mathbf{r}(t)$ is just $(\cos t)\mathbf{a} + (\sin t)\mathbf{b}$.

Now on to the integral. We have $\mathbf{F}(\mathbf{r}(t)) = (-\cos t - 1)\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -\sin t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{2}\cos t\,\mathbf{k}$. The dot product of these is $(\sin t\cos t + \sin t) + \sin t\cos t + \sqrt{2}\cos t$, and $2\sin t\cos t$

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

simplifies to $\sin 2t$, so we integrate

$$\int_{t=0}^{\pi} (\sin 2t + \sin t + \sqrt{2}\cos t)\,\mathrm{d}t = -\frac{1}{2}\cos 2t - \cos t + \sqrt{2}\sin t\Big|_{t=0}^{\pi} = 2.$$

(The only nonzero contribution is from the $-\cos t$ term, which is 1 at $t = \pi$ and -1 at $t = 0$.)

Now let's try the much easier integral along the path in Figure 1b, which is parameterized by $\mathbf{r}(t) = (-t, t, 0)$, where $t \in [-1, 1]$. We have $\mathbf{F}(\mathbf{r}(t)) = (t-1)\,\mathbf{i} - t\,\mathbf{j} + \mathbf{k}$, $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (-1, 1, 0)$, and $\mathbf{F}(\mathbf{r}(t)) = (1-t) - t = 1 - 2t$. Integrating,

$$\int_{t=-1}^{1} (1-2t)\,\mathrm{d}t = t - t^2\Big|_{t=-1}^{1} = 1 - (-1) = 2.$$

This could be a coincidence, but I promise you it's not: the vector field $\mathbf{F} = (y-1)\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}$ is actually a conservative vector field!

Here are a few questions to see how we can apply this assumption.

Q: Assuming $\mathbf{F}$ is a conservative field, what should the integral of $\mathbf{F}$ be along the path in Figure 1c?

A: The line integral of $\mathbf{F}$ only depends on the start and end, but the start and end of the semicircle in Figure 1c are *swapped* from the start and end of the semicircle in Figure 1a! Since the integral along the path in Figure 1a was 2, the integral of the reverse of that path should be -2. Then the integral over the path in Figure 1c should also be -2, since it has the same start and end as the reverse of the first path.

Q: Assuming $\mathbf{F}$ is a conservative field, what should the integral of $\mathbf{F}$ be along a path which completes the semicircle in FIgure 1a to a whole circle?

A: This integral would be the sum of two integrals: the integral along the curve in Figure 1a (call that curve C_1) and the integral along the curve in Figure 1c (call that curve C_2). We get

$$\int_{C_1 \cup C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2 + (-2) = 0.$$

This argument can be generalized to say that path independence is in fact *the same property* as giving 0 along any closed loop. We can think of any closed curve as a path from point A to point B, followed by a path from point B back to point A, and the integral along the second path will be the negative of the integral of the first path, by path independence.

This observation is important, and you should remember it.

1.2 Shortcuts using path independence

Can we simplify calculations somehow with the knowledge that the vector field $\mathbf{F}$ is conservative? Well, here's one thing we can try.

Define a scalar function $f\colon \mathbb{R}^3 \to \mathbb{R}$ as follows: to evaluate f at a point (a, b, c), let C be any path we like from $(0, 0, 0)$ to (a, b, c), and let $f(a, b, c) = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. By path independence, the value of $f(a, b, c)$ does not depend on the choice of C: it really is just a function of the point (a, b, c).

One way to find a formula for $f(a,b,c)$ is to pick a particular simple path. Forexample, let $\mathbf{r}(t) = (at, bt, ct)$, where $t \in [0,1]$. Then for the same example $\mathbf{F} = (y-1)\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}$, we get $\mathbf{F}(\mathbf{r}(t)) = (bt-1)\,\mathbf{i} + at\,\mathbf{j} + \mathbf{k}$, $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, so $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = a(bt-1) + b(at) + c(1) = 2abt - a + c$, and

$$f(a,b,c) = \int_{t=0}^{1} (2abt - a + c)\,\mathrm{d}t = abt^2 - at + ct\Big|_{t=0}^{1} = ab - a + c.$$

Now let's see how this helps us.

Suppose I take an impossibly complicated curve C such as the one parameterized by

$$\mathbf{r}(t) = (e^{3t}\cos(\pi t^2), \sqrt{3t^2+1}, (t+3)^{1+t/2}), \qquad t \in [0,1].$$

This curve starts at $\mathbf{r}(0) = (1,1,3)$ and ends at $\mathbf{r}(1) = (-e^3, 2, 8)$. We *could* replace C by some other curve with the same start and end, and take an integral along that curve, but there's an easier way.

Let C_1 be any curve that begins at $(0,0,0)$ and ends at $(1,1,3)$. Then $C_1 \cup C$ is a curve that begins at $(0,0,0)$ and ends at $(e^3, 2, 8)$. Therefore

$$f(1,1,3) = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \qquad \text{and} \qquad f(e^3,2,8) = \int_{C_1 \cup C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

By subtracting, we get

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(e^3,2,8) - f(1,1,3) = (2e^3 - e^3 + 8) - (1 - 1 + 3) = e^3 + 6.$$

Once we have a formula for the function f, we can use it to find *any* line integral of $\mathbf{F}$ along *any* curve! If the curve starts at point A and ends at point B, we will get $f(B) - f(A)$ as the answer, by the same argument as above.

One more question:

Q: Is there a connection between $\mathbf{F} = (y-1)\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}$ and $f(x,y,z) = xy - x + z$?

A: Yes! In fact, $\mathbf{F} = \boldsymbol{\nabla} f$, the gradient field of f. This is not a coincidence.

2 The fundamental theorem of line integrals

The not-a-coincidence above can be stated as a theorem:

Theorem 2.1 (Fundamental theorem of line integrals)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with a continuous gradient vector $\mathbf{F} = \boldsymbol{\nabla} f$. Let C be a smooth curve in $\mathbb{R}^n$ from point A to point B. Then*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(B) - f(A).$$

This theorem assumes $\mathbf{F}$ and f are nice everywhere. It's okay if they are only well-behaved on some subset of $\mathbb{R}^n$, but with important caveats. In general, if $\mathbf{F}$ is not defined everywhere, we should worry! More on this later.

Proof. (This is a proof over $\mathbb{R}^3$, though it's the same over $\mathbb{R}^2$ or over $\mathbb{R}^n$ for any n.)

The chain rule for functions of several variables says that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t), z(t)) = \boldsymbol{\nabla} f(x(t), y(t), z(t)) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathbf{k} \right).$$

In particular, if our curve C is parameterized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, where $t \in [a, b]$, then this chain rule says that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t)) = \boldsymbol{\nabla} f(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Therefore

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t=a}^{b} \boldsymbol{\nabla} f(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_{t=a}^{b} \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\,\mathrm{d}t.$$

By the fundamental theorem of calculus, this is equal to $f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A)$. □

This theorem tells us that gradient fields are always conservative, but the reverse is also always true. Whenever a vector field is conservative, there is only one explanation: it is the gradient field of some scalar function f.

To prove this in general, we reason in the same way as we did in our example. We can define a function f in terms of line integrals from $(0, 0, 0)$ to a point, and conclude that the integral of $\mathbf{F}$ over a curve from point A to point B equals $f(B) - f(A)$ as we did before. From there, we stand back and watch as $\mathbf{F}$ turns out to be the gradient $\boldsymbol{\nabla} f$.

Rather than go through the proof in full detail, here is some intuition for it. What is the partial derivative $\frac{\partial f}{\partial x}$? It is defined by the limit

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

If we let C be a straight-line path from (x, y, z) to $(x + h, y, z)$, then the numerator inside the limit is equal to $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

However, if we're going along a very short straight-line path with displacement vector $h\,\mathbf{i}$, then (up to lower-order terms that disappear in the limit as $h \to 0$) the line integral is just $\mathbf{F}(x, y, z) \cdot h\,\mathbf{i}$, which is just h times the x-component of $\mathbf{F}$. Dividing by h and taking the limit as h goes to 0, we conclude that $\frac{\partial f}{\partial x}$ *is* the x-component of $\mathbf{F}$. Repeating the same argument for y and z, we conclude that $\mathbf{F} = \boldsymbol{\nabla} f$.

As a side note, there is one formulation of the fundamental theorem of line integrals that's particularly easy to state. The line integral of $\boldsymbol{\nabla} f$ can be written as

$$\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = \int_C \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z,$$

and we have already (when discussing multivariate substitutions) used the notation $\mathrm{d}f$ to mean the differential form $\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$, which is precisely what we're integrating above. So we can state the equation Theorem 2.1 as:

$$\int_C \mathrm{d}f = f(B) - f(A).$$