Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 12: A tricky vector field

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1 Differential forms

We think of the circulation and flux integrals as integrals of vector fields. However, we can also think of them as integrals of differential forms, via the definition

$$\int_C M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = \int_{t=a}^{b} \left(M(x(t),y(t))\frac{\mathrm{d}x}{\mathrm{d}t} + N(x(t),y(t))\frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t$$

(where C is parameterized as $(x(t),y(t))$ for $t \in [a,b]$). From this point of view, flux and circulation integrals are the same object. For example, the circulation integral of $e^{x+y}\,\mathbf{i} + \sqrt{x^2+y^2}\,\mathbf{j}$ around C and the flux integral of $\sqrt{x^2+y^2}\,\mathbf{i} - e^{x+y}\,\mathbf{j}$ across C are actually the same integral,

$$\int_C e^{x+y}\,\mathrm{d}x + \sqrt{x^2+y^2}\,\mathrm{d}y = \int_C \sqrt{x^2+y^2}\,\mathrm{d}y - (-e^{x+y})\,\mathrm{d}x.$$

Now let's talk about how gradient fields and the component test interact with differential forms. Today, we will mostly focus on examples in $\mathbb{R}^2$, but all of this generalizes to $\mathbb{R}^3$ and even beyond, to $\mathbb{R}^n$.

1.1 Exact differential forms

For a differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$, we define $\mathrm{d}f$ to be the differential form $\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$: the differential equivalent of the gradient $\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$. We've already seen that in this form, the fundamental theorem of line integrals has a particularly clean statement. It states that when curve C starts at point $\mathbf{a}$ and ends at point $\mathbf{b}$,

$$\int_C \mathrm{d}f = f(\mathbf{b}) - f(\mathbf{a}).$$

A differential form $M\,\mathrm{d}x + N\,\mathrm{d}y$ is called **exact** if $M\,\mathrm{d}x + N\,\mathrm{d}y = \mathrm{d}f$ for some f. Sorry, this is yet another term alongside "conservative" and "gradient" and "path-independent" for approximately the same concept. We won't use it much apart from today, but you might encounter it elsewhere.

Anyway, exact differential forms are useful because their integrals are path-independent, and their integrals along a closed curve are always equal to 0.

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

1.2 Closed differential forms

We can also define the derivative (formally, the **exterior derivative**) of a differential form. This is defined by the rule

$$\mathrm{d}(M\,\mathrm{d}x + N\,\mathrm{d}y) = \mathrm{d}M \wedge \mathrm{d}x + \mathrm{d}N \wedge \mathrm{d}y,$$

in terms of wedge products, which we briefly saw earlier.[2] We simplify this wedge product by the rules that $\mathrm{d}y \wedge \mathrm{d}x = -(\mathrm{d}x \wedge \mathrm{d}y)$ and $\mathrm{d}x \wedge \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}y = 0$. So, for example,

$$\begin{aligned}
\mathrm{d}(x^2y\,\mathrm{d}x + e^{x-y}\,\mathrm{d}y) &= (2xy\,\mathrm{d}x + x^2\,\mathrm{d}y) \wedge \mathrm{d}x + (e^{x-y}\,\mathrm{d}x - e^{x-y}\,\mathrm{d}y) \wedge \mathrm{d}y \\
&= 2xy \cdot 0 + x^2\,\mathrm{d}y \wedge \mathrm{d}x + e^{x-y}\,\mathrm{d}x \wedge \mathrm{d}y - e^{x-y} \cdot 0 \\
&= (-x^2 + e^{x-y})\,\mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

(This multiple of $\mathrm{d}x \wedge \mathrm{d}y$ is also a differential form, but a differential form of **degree** 2, rather than 1. We could integrate it over a 2-dimensional region, rather than a 1-dimensional region.)

In general, we end up with the identity

$$\mathrm{d}(M\,\mathrm{d}x + N\,\mathrm{d}y) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y.$$

Why is this important to us right now? Because the component test says that if a vector field $M\,\mathbf{i} + N\,\mathbf{j}$ is a gradient field, then $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$; in other words, $\mathrm{d}(M\,\mathrm{d}x + N\,\mathrm{d}y) = 0$. In the language of differential forms: if $M\,\mathrm{d}x + N\,\mathrm{d}y$ is exact, then $\mathrm{d}(M\,\mathrm{d}x + N\,\mathrm{d}y) = 0$.

We call a differential form ϕ **closed** if $\mathrm{d}\phi = 0$. That is, another way to phrase the component test is "all exact differential forms are closed".

Are all closed differential forms exact? In other words, does the component test guarantee that a vector field is conservative? This is the question we will investigate today.

2 The $\mathrm{d}\theta$ differential form

In polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$. We also often write the identity $r = \sqrt{x^2 + y^2}$. We rarely write θ in terms of x and y, and that's because doing so is a bit awkward.

Dividing y by x, we get $\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$. It is not *precisely* true that $\theta = \arctan\frac{y}{x}$: this fails to distinguish the point (x, y) from the point $(-x, -y)$, which have the same value of $\frac{y}{x}$ but have angles that differ by π. Usually, θ ranges from 0 to 2π, but the output of arctan is taken to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Therefore it's valid to say that $\theta = \arctan\frac{y}{x}$ in the first quadrant; outside that range, we should add or subtract a constant.

But derivatives don't care about that constant, so we can take the derivatives

$$\frac{\partial\theta}{\partial x} = \frac{1}{1 + (y/x)^2} \cdot -\frac{y}{x^2} = \frac{-y}{x^2 + y^2}, \qquad \frac{\partial\theta}{\partial y} = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

and have these be valid for any value of θ. Today, we will look closely at the vector field

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}.$$

[2] In $\mathbb{R}^3$, we define $\mathrm{d}(M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z)$ as $\mathrm{d}M \wedge \mathrm{d}x + \mathrm{d}N \wedge \mathrm{d}y + \mathrm{d}P \wedge \mathrm{d}z$, continuing the pattern.

This corresponds to the differential form $\mathrm{d}\theta = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$.

Does $\mathbf{F}$ pass the component test? In other words, is $\mathrm{d}\theta$ closed? You will not be surprised to learn that the answer is yes. Explicitly,

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right).$$

Is the vector field $\mathbf{F}$ conservative? In other words, is $\mathrm{d}\theta$ exact? This seems like a silly question. Conservative fields are gradient fields, and $\mathbf{F}$ is the gradient of $\theta(x, y)$. Exact differential forms are those we can write as $\mathrm{d}f$ for some f, and $\mathrm{d}\theta$ is already written that way.

The problem is that θ is not exactly a function. Or at least, it's not a continuous function. If you have θ range from 0 to 2π, then as you go around the unit circle counterclockwise, θ will keep increasing and increasing until it suddenly drops back down to 0. It will be continuous everywhere except for the ray $\{(x, 0) : x \geq 0\}$. We can move the cutoff point if we like, but θ will always have a discontinuity along our cutoff point, and it will *never* be continuous at $(0, 0)$.

The answer to whether $\mathbf{F}$ is conservative, or whether $\mathrm{d}\theta$ is exact is: **it depends on the domain**.

For example, let D be the set $\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$: all points in the plane, *except* for the nonnegative x-axis. On this domain, we can define a continuous function $\theta(x, y)$ which is always between 0 and 2π, and then we will have $\mathbf{F} = \boldsymbol{\nabla}\theta$. On this domain, we can define a path C in the shape of the letter C: the path parameterized by $\mathbf{r}(t) = (\cos t, \sin t)$, where $t \in [0.01, 6.28]$. What will we get when we integrate

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathrm{d}\theta?$$

By the fundamental theorem of line integrals, it will be $\theta(\mathbf{r}(6.28)) - \theta(\mathbf{r}(0.01))$, which simplifies to $6.28 - 0.01 = 6.27$. For all curves inside this set D, integrating $\mathbf{F}$ will behave as nicely as we want. For example, for any other curve C' that begins at $(\cos 0.01, \sin 0.01)$ and ends at $(\cos 6.28, \sin 6.28)$, the integral of $\mathbf{F}$ along C' will also be 6.27. If we take a closed loop entirely contained in the domain D, the integral of $\mathbf{F}$ around that loop will be 0.

But now, let C be the entire unit circle: a curve that's *not* contained in D (because it passes through the point $(1, 0) \notin D$). What will

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

be? Well, our previous curve had almost the same endpoints, and had an integral of almost 2π, so you probably won't be surprised to get the same answer here. We can do it the hard way, as well:

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \\
&= \int_{t=0}^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t)\right) \mathrm{d}t \\
&= \int_{t=0}^{2\pi} \left(\frac{(-\sin t)^2 + (\cos t)^2}{\cos^2 t + \sin^2 t}\right) \mathrm{d}t = \int_{t=0}^{2\pi} 1\,\mathrm{d}t = 2\pi.
\end{aligned}$$

This is the integral of $\mathbf{F}$ around a closed loop, and it's not 0. Similarly, we can find an example where $\mathbf{F}$ fails to be path-independent. Go from $(1,0)$ to $(-1,0)$ around the top half of the unit circle, and you get π, but go from $(1,0)$ to $(-1,0)$ around the bottom of the unit circle, and you get $-\pi$.

Therefore, on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$ (the biggest domain where we can define $\mathbf{F}$: at $(0,0)$, we'd be dividing by 0) the vector field $\mathbf{F}$ is not conservative, and $\mathrm{d}\theta$ is not exact.

3 Integrating around undefined points

Do we constantly have to be worried, when we're dealing with vector fields that "look" conservative, that they'll misbehave in the same way?

For the most part, the answer is no. If a vector field is defined on all of $\mathbb{R}^2$ (or in general, on all of $\mathbb{R}^n$), and it passes the component test, then it is in fact conservative. In other words, a differential form defined on all of $\mathbb{R}^n$ is exact if and only if it is closed.

A general condition is the following:

Theorem 3.1. *Let $D \subseteq \mathbb{R}^n$ be an open,* ***simply connected*** *domain: informally, D does not include its boundary, and has no holes in it. Let $\mathbf{F}$ be a vector field defined on all of D.*

Then $\mathbf{F}$ is conservative, and is equal to ∇f for some function $f\colon D \to \mathbb{R}$, if and only if it passes the component test.

In other words: if D is open and simply connected, then a differential form defined on D is exact if and only if it is closed. (But we may only use this result when integrating over curves contained entirely in D.)

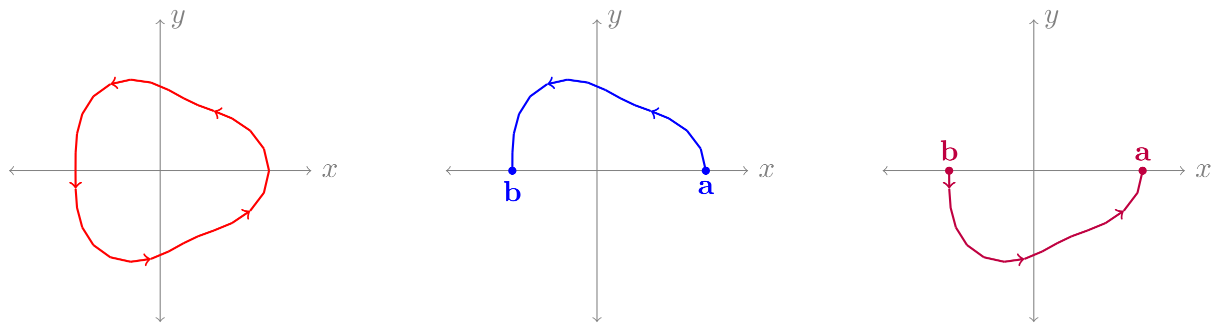
The troubling thing about $\mathbf{F} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$ is that the point $(0,0)$ was causing us trouble even when integrating along a curve that didn't get *close* to $(0,0)$. Somehow, our line integral sensed that the undefined point was lurking inside the curve, and decided to misbehave!

3.1 Partial potential functions

The vector field $\mathbf{F} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$ (corresponding to the differential form $\mathrm{d}\theta$) still has some very nice features even though it's not conservative. By the component test, it has a potential function on every open, simply connected domain not containing $(0,0)$—it's just that these potential functions may not all agree. We also know what those potential functions are: they are various possible measurements of the angle θ.

So suppose I give you some wonky closed curve around $(0,0)$, such as the one in Figure 1a. Can we use our knowledge to integrate $\mathbf{F}$ around this path?

What we can do—what we can always do—is break up our closed loop into two segments. Let's let point $\mathbf{a}$ be the point where our loop crosses the positive x-axis, and let $\mathbf{b}$ be the point where it crosses the negative x-axis. We first integrate along the top half, from $\mathbf{a}$ to $\mathbf{b}$, and then along the bottom half, from $\mathbf{b}$ back to $\mathbf{a}$.



(a) A closed loop around $(0, 0)$ (b) The top half of the loop (c) The bottom half of the loop

Figure 1: Integrating $\mathbf{F} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$ in a closed loop around $(0, 0)$

What can we do for the first loop (shown in Figure 1b)? Well, let $\theta_1(x, y)$ be the counterclockwise angle that the vector from $(0, 0)$ to (x, y) makes with the *negative y-axis*. This is a weird choice, but it's equal to our usual θ coordinate up to a constant, so $\mathrm{d}\theta_1$ will be the same as $\mathrm{d}\theta$ anywhere they're both defined. Also, if we let $D_1 = \mathbb{R}^2 \setminus \{(0, y) : y \le 0\}$ (excluding the negative y-axis itself), then θ_1 is a continuous potential function for $\mathbf{F}$ on the domain D_1.

Therefore, by the fundamental theorem of line integrals, the integral of $\mathbf{F}$ along the segment of our curve from $\mathbf{a}$ to $\mathbf{b}$ is given by $\theta_1(\mathbf{b}) - \theta_1(\mathbf{a})$, which is $\frac{3\pi}{2} - \frac{\pi}{2} = \pi$.

For the second loop (shown in Figure 1c), we do almost the same thing, but we define $\theta_2(x, y)$ to be the counterclockwise angle with the *positive* y-axis. Then θ_2 is a continuous potential function for $\mathbf{F}$ on the domain $D_2 = \mathbb{R}^2 \setminus \{(0, y) : y > 0\}$, and the bottom half of our curve is contained entirely inside D_2. The integral along the segment of our curve from $\mathbf{b}$ back to $\mathbf{a}$ is given by $\theta_2(\mathbf{a}) - \theta_2(\mathbf{b})$, which is also $\frac{3\pi}{2} - \frac{\pi}{2} = \pi$.

Therefore the integral going all the way around the origin is still $\pi + \pi = 2\pi$, even for this weird curve.

3.2 The gravity field

Newton's law of universal gravitation states a mass of M at the origin pulls on an object at (x, y) of mass m with the force

$$\mathbf{F} = -\frac{GmM}{(x^2 + y^2)^{3/2}}(x\,\mathbf{i} + y\,\mathbf{j}).$$

Here, G, m, and M are constants, and we're mathematicians, so from now on let's just say they're all 1.

(Formally, we should be doing this in three dimensions, but that's not harder, it's just more time-consuming. Also, you may be surprised by the exponent of $3/2$. Here's, the explanation: the magnitude $\|\mathbf{F}\|$ is $\frac{GmM}{x^2+y^2} = \frac{GmM}{r^2}$, it's just that the direction is given by the unit vector $-\frac{x\,\mathbf{i}+y\,\mathbf{j}}{\sqrt{x^2+y^2}}$.)

This force field is also undefined at $(0, 0)$: we're also dividing by 0 there. So should we be just as worried about what's going on over here?

One answer is that we're safe as soon as we find a potential function. Here, the vector field

$$\mathbf{F} = -\frac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2+y^2)^{3/2}}$$

has potential function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$.

This is still undefined at $(0,0)$, but it is defined and continuous everywhere else, unlike $\theta(x,y)$. So we are allowed to use it whenever we're integrating along a curve that doesn't *actually pass through* the origin. (And if our curve does pass through the origin, we have bigger problems.)

Suppose, however, that for whatever reason, finding a potential function is beyond us. It involves taking antiderivatives, after all; sometimes those are hard. We can check that $\mathbf{F}$ passes the component test, but we've learned the hard way that with an undefined point at $(0,0)$, integrating around $(0,0)$ could still go wrong.

Well, if $\mathbf{F}$ passes the component test, we know that it's conservative on open, simply connected domains where it is defined. How can we make use of this?

To begin with, what this means is that for two curves C_1 and C_2 with the same endpoints, path independence holds as long as the space between C_1 and C_2 does not contain $(0,0)$, where $\mathbf{F}$ is undefined. Another way to put it is that we have path independence if you can imagine deforming C_1 to turn it into C_2, without any intermediate steps that pass through the origin.

We can deal with the origin by a single "test integral". Suppose we integrate $\mathbf{F}$ around the unit circle: the set of points where $x^2+y^2=1$. If C is the unit circle, we can say that

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = 0$$

because $\mathbf{F}\cdot\mathbf{T} = 0$ at every point of the unit circle: the force $\mathbf{F}$ always points towards the origin, perpendicular to the tangent vector.

If a single closed curve around the undefined point integrates to 0, then *all* of them must integrate to 0. Essentially, you can deform any closed curve around $(0,0)$ into the unit circle. Then, path independence on some simply connected regions not containing $(0,0)$ will guarantee that we haven't changed the integral while doing this deformation.