Math 3204: Calculus IV[1] — Mikhail Lavrov

Lecture 17: Surface area

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1 Review of cross products

Let's begin with a review of cross products. Given vectors $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, we define the cross product $\mathbf{a} \times \mathbf{b}$ by the formula

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}.$$

As an algebraic consequence of this definition, if we take the dot product of $\mathbf{a} \times \mathbf{b}$ with a third vector $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$, that's equivalent to replacing $\mathbf{i}, \mathbf{j}, \mathbf{k}$ by c_1, c_2, c_3 in the formula, which gives the determinant of a 3×3 matrix with rows $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (Technically, the rows end up in the order $\mathbf{c}, \mathbf{a}, \mathbf{b}$, but that determinant is the same.)

The cross product of $\mathbf{a}$ and $\mathbf{b}$ has also two important geometric properties.

1. $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. This follows from the algebraic fact above: if we take the dot product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$, that's equal to the determinant of a matrix with rows $\mathbf{a}$, $\mathbf{a}$, and $\mathbf{b}$. The determinant of a matrix with repeated rows is 0, so the dot product is 0, which means $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$. (We can use a similar argument to show that it is perpendicular to $\mathbf{b}$.)

2. The magnitude $\|\mathbf{a} \times \mathbf{b}\|$ can be expressed in two equivalent ways. First of all, $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Second, $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$.

   This last fact is what we'll use the cross product for today—but the other details will be relevant later.

The second property tells us the magnitude of $\mathbf{a} \times \mathbf{b}$, and the first property *almost* tells us the direction. There is one ambiguity: if $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, then $-\mathbf{c}$ is another perpendicular vector with the same magnitude.

The determinant formula for $\mathbf{a} \times \mathbf{b}$ picks the vector that follows the **right-hand rule**:

> "If you place your right hand on the **ab**-plane in the direction of $\mathbf{a}$ with your fingers curling toward $\mathbf{b}$, then your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$."

This can require great physical contortion for some cross products; it's best if you're good at imagining your hand in various positions without actually moving it.

As a corollary of this rule, or the determinant formula, the cross product is **anti-commutative**: $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$.

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

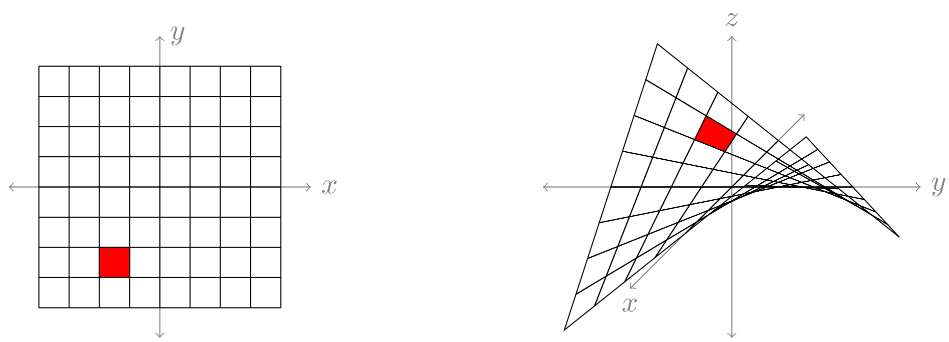
2 Surface area

We needed to review the cross product today because we will use it for our first surface integral: the surface area integral.

As our example, let's take the surface with equation $z = xy$, with $-1 \le x \le 1$ and $-1 \le y \le 1$. (A rotated version of this surface appeared in an example in the previous lecture.) We can set $x = u$ and $y = v$ to parameterize this surface, getting

$$\mathbf{r}(u,v) = (u, v, uv) \qquad (u,v) \in [-1,1] \times [-1,1].$$

You can see a picture of this surface in Figure 1b; it is called a "hyperboloid" and is notable for the saddle-shaped bend it has near the origin.



(a) The domain of the parameterization

(b) The surface $z = xy$

Figure 1: Visualizing the scalar line integral of a 2-variable function

To find the area of a surface, we will start with a discrete approximation and then take a limit, as usual.

If our parameterization has domain $[a,b] \times [c,d]$ in the uv-plane, we can divide this domain into many square (or rectangular) cells, as shown in Figure 1a. Each of these square cells is mapped by $\mathbf{r}$ to a tiny piece of the surface; the cells highlighted in red in Figure 1 are an example of this. Even this tiny piece of the surface has some curvature, but as an approximation, we will pretend that it is flat.

If the cell in the uv-plane is $[u, u+\Delta u] \times [v, v+\Delta v]$, then $\mathbf{r}$ sends that cell to an approximate parallelogram in $\mathbb{R}^3$, with one side parallel to $\mathbf{a} = \mathbf{r}(u+\Delta u, v) - \mathbf{r}(u,v)$ and one side parallel to $\mathbf{b} = \mathbf{r}(u, v+\Delta v) - \mathbf{r}(u,v)$. The area of that parallelogram is $\|\mathbf{a} \times \mathbf{b}\|$, which we can rewrite as

$$\left\| \frac{\mathbf{r}(u+\Delta u, v) - \mathbf{r}(u,v)}{\Delta u} \times \frac{\mathbf{r}(u, v+\Delta v) - \mathbf{r}(u,v)}{\Delta v} \right\| \Delta u \, \Delta v. \tag{1}$$

(A fact we're implicitly using here is that the cross product is linear: for a constant $k \in \mathbb{R}$, $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$. This is true because the determinant is linear.)

In the limit as $\Delta u \to 0$ and $\Delta v \to 0$, the two inputs to the cross product of (1) approach the partial derivatives $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$. If we sum this expression over all the small cells making up the domain

$[a, b] \times [c, d]$, we get an integral

$$\int_{u=a}^{b} \int_{v=c}^{d} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, \mathrm{d}v \, \mathrm{d}u.$$

This is the integral we will use to find surface areas!

In our example, $\mathbf{r}(u, v) = (u, v, uv)$, so $\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + v\,\mathbf{k}$ and $\frac{\partial \mathbf{r}}{\partial v} = \mathbf{j} + u\,\mathbf{k}$. Their cross product is

$$(\mathbf{i} + v\,\mathbf{k}) \times (\mathbf{j} + u\,\mathbf{k}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{bmatrix} = -v\,\mathbf{i} - u\,\mathbf{j} + \mathbf{k}.$$

Therefore $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\| = \sqrt{(-v)^2 + (-u)^2 + 1} = \sqrt{1 + u^2 + v^2}$, and the surface area of the hyperboloid is given by

$$\int_{u=-1}^{1} \int_{v=-1}^{1} \sqrt{1 + u^2 + v^2} \, \mathrm{d}v \, \mathrm{d}u.$$

After some trivial computations[2], this integral simplifies to $\frac{4}{3}(\sqrt{3} + \ln(7 + 4\sqrt{3})) - \frac{2}{9}\pi$.

3 Spherical surface area the hard way and the easy way

Let's use this technique to find the surface area of a sphere. Center our sphere at $(0, 0, 0)$ and give it radius ρ. We can give it a parameterization based on spherical coordinates:

$$\mathbf{r}(\phi, \theta) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi), \qquad (\phi, \theta) \in [0, \pi] \times [0, 2\pi].$$

(Though ρ here is just the spherical coordinate ρ, in this parameterization it is held constant; it is not one of the variables.)

We can do this the hard way, straight from the definition:

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \rho \cos\theta \cos\phi & \rho \sin\theta \cos\phi & -\rho \sin\phi \\ -\rho \sin\theta \sin\phi & \rho \cos\theta \sin\phi & 0 \end{bmatrix}$$

which simplifies to

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \rho^2 \cos\theta \sin^2\phi \,\mathbf{i} + \rho^2 \sin\theta \sin^2\phi \,\mathbf{j} + \rho^2 \sin\phi \cos\phi \,\mathbf{k}.$$

A lot of simplification happens when we take the norm of this vector—I won't spoil the answer just yet. We can get there in a roundabout way, by using a trick.

Let's change our mind and think of $\mathbf{r}$ as a function of ρ as well as ϕ and θ: now it is no longer a surface parameterization, but just the spherical coordinate substitution. From earlier in these notes, we know that taking a dot product with a cross product just gives us a determinant of

[2] In this case, by "trivial", I mean that it doesn't take Mathematica too long to do them for me, because I'm definitely not taking this integral by hand.

the matrix formed by three vectors. If we take the dot product with $\frac{\partial \mathbf{r}}{\partial \rho}$, then we're taking the determinant of the matrix of partial derivatives—the Jacobian determinant! Therefore

$$\frac{\partial \mathbf{r}}{\partial \rho} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right) = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi.$$

But what is this dot product, and what is $\frac{\partial \mathbf{r}}{\partial \rho}$? Each component of $\mathbf{r}$ is linear in ρ, so taking $\frac{\partial}{\partial \rho}$ is just dividing $\mathbf{r}$ by ρ, turning a vector of length ρ into a vector of length 1.

Moreover, it turns out that $\frac{\partial \mathbf{r}}{\partial \rho}$ is parallel to $\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}$; that's because it's perpendicular to both $\frac{\partial \mathbf{r}}{\partial \theta}$ and $\frac{\partial \mathbf{r}}{\partial \phi}$. Those two partial derivatives are both tangent to the surface of the sphere; meanwhile, since $\mathbf{r}$ is a radial vector, and $\frac{\partial \mathbf{r}}{\partial \rho}$ is proportional to it, it is perpendicular to the surface of the sphere.

Therefore, we have

$$\left| \frac{\partial \mathbf{r}}{\partial \rho} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right) \right| = \left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\| \cdot \left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\|.$$

We already mentioned that $\|\frac{\partial \mathbf{r}}{\partial \rho}\| = 1$, so we finally get

$$\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \left| \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \right| = \rho^2 \sin \phi.$$

What this means is that just like $\rho^2 \sin \phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$ is the volume element when integrating in spherical coordinates, $\rho^2 \sin \phi \, \mathrm{d}\phi \, \mathrm{d}\theta$ is also the area element when integrating over the surface of a sphere of radius ρ. In particular,

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \rho^2 \sin \phi \, \mathrm{d}\phi \, \mathrm{d}\theta$$

is the integral that gives us the area of a sphere of radius ρ. Integrating $\sin \phi$ from 0 to π gives 2, which we multiply by ρ^2 and then by 2π (the length of the domain of θ), so the surface area of the sphere is $4\pi\rho^2$.