Math 3204: Calculus IV[1] — Mikhail Lavrov

Lecture 18: More surface area

October 18, 2023 — *Kennesaw State University*

1 Surface area for simple parameterizations

I have already said that "wacky-domain" surface parameterizations $\mathbf{r}\colon D \to \mathbb{R}^3$ where D *does not* have the form $[a,b]\times[c,d]$ are usually to be avoided.

About the only reason to use them is in a case where the surface we want to describe is given by an equation $z = h(x,y)$, with some bounds on x and y that we can specify by saying that (x,y) lies in some subset $D \subseteq \mathbb{R}^2$. In this case, the benefit of the wacky domain outweighs the cos: we can write

$$\mathbf{r}(x,y) = (x,y,h(x,y)), \qquad (x,y) \in D$$

and get a very simple parameterization.

In fact, we can figure out a general formula for the surface area of such a surface:

Proposition 1.1. *Let a surface S be given by the equation $z = h(x,y)$ with the constraint that $(x,y)\in D$. Then the area of S can be found using the integral*

$$\iint_D \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. In this special case, using the parameterization $\mathbf{r}(x,y) = (x,y,h(x,y))$ where $(x,y)\in D$, we can simplify the factor $\|\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\|$. The cross product here is going to be

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial h}{\partial x} \\ 0 & 1 & \frac{\partial h}{\partial y}\end{bmatrix} = -\frac{\partial h}{\partial x}\,\mathbf{i} - \frac{\partial h}{\partial y}\,\mathbf{j} + \mathbf{k}$$

and the norm of this vector is precisely $\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}$. $\square$

We saw an example of this in the previous lecture, with the surface $z = xy$ bounded by $-1 \le x \le 1$ and $-1 \le y \le 1$. In this case, the integrand becomes $\sqrt{x^2+y^2+1}$.

2 Implicit surface area

Now let's conside a very slightly more general case. Suppose that our parameterization still *could* be given by an equation $z = h(x,y)$, where $(x,y)\in D$, *but* we are not given h explicitly; we are given the equation of the surface in the implicit form $f(x,y,z) = 0$.

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

For example, suppose that we are (once again) looking at the top half of the unit sphere: the surface given by $x^2 + y^2 + z^2 = 1$, where $z \geq 0$. We could, if we so chose, parameterize this by

$$\mathbf{r}(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), \qquad x^2 + y^2 \leq 1.$$

However, if we did this, we'd have to take so many derivatives of square roots, and that's a pain. Can we avoid it?

Let's set out some ground rules first. Even if we manage to avoid dealing with the explicit form $z = h(x, y)$, such a form *must still exist.* In other words, for all $(x, y) \in D$, there must be a *unique* z such that $f(x, y, z) = 0$. Geometrically, this means that our surface must pass the "vertical line" test: any line parallel to the z-axis can intersect the surface at most once.

(The hemisphere given by $x^2 + y^2 + z^2 = 1$ with the constraint $z \geq 0$ has this property, but the entire sphere does not!)

How can we find $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ without taking the derivative of h? Well, we can start from the equation

$$f(x, y, h(x, y)) = 0.$$

Taking the partial derivative with respect to x gives us

$$\frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot 0 + \frac{\partial f}{\partial z} \cdot \frac{\partial h}{\partial x} = 0 \iff \frac{\partial h}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}$$

(using the multivariate chain rule). Similarly, taking the partial derivative of $f(x, y, h(x, y)) = 0$ with respect to y gives us

$$\frac{\partial f}{\partial x} \cdot 0 + \frac{\partial f}{\partial y} \cdot 1 + \frac{\partial f}{\partial z} \cdot \frac{\partial h}{\partial y} = 0 \iff \frac{\partial h}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

Therefore

$$\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1} = \sqrt{\left(\frac{\partial f/\partial x}{\partial f/\partial z}\right)^2 + \left(\frac{\partial f/\partial y}{\partial f/\partial z}\right)^2 + 1} = \frac{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}{\left|\frac{\partial f}{\partial z}\right|}.$$

We can write the numerator of this fraction as $\|\boldsymbol{\nabla} f\|$, and the denominator as $|\boldsymbol{\nabla} f \cdot \mathbf{k}|$. The result is the following formula:

Proposition 2.1. *Let a surface S be given by the equation $f(x, y, z) = 0$ with the constraint that $(x, y) \in D$, and any additional constraints that ensure that for every pair $(x, y) \in D$, there is a unique z such that (x, y, z) lies on S. Then the area of S can be found using the integral*

$$\iint_D \frac{\|\boldsymbol{\nabla} f\|}{|\boldsymbol{\nabla} f \cdot \mathbf{k}|} \, \mathrm{d}x \, \mathrm{d}y.$$

3 Examples, quibbles, and generalizations

Let's see an example of this formula in use, and then discuss what could go wrong here.

For the area of the hemisphere we were looking at, if the equation has the form $f(x, y, z) = 0$, then $f(x, y, z) = x^2 + y^2 + z^2 - 1$. This gives us $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$, with two consequences for us:

- The norm $\|\nabla f\|$ is $\sqrt{(2x)^2 + (2y)^2 + (2z)^2} = 2\sqrt{x^2 + y^2 + z^2}$.
- $|\nabla f \cdot \mathbf{k}| = |2z|$.

It seems that we get the integral

$$\iint_D \frac{2\sqrt{x^2+y^2+z^2}}{|2z|}\,\mathrm{d}x\,\mathrm{d}y$$

where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

There's something awkward about this integral: it still has z in it. We can't take the integral like that! If we're taking an integral with respect to x and y, we had better have no other variables in that integral.

This means that we can't escape solving for z entirely. In this case, the numerator $2\sqrt{x^2+y^2+z^2}$ can just be replaced with 2, since $x^2+y^2+z^2 = 1$ everywhere on the surface S. In the denominator, we'll have to suffer the indignity of having to rewrite z as $\sqrt{1-x^2-y^2}$. The resulting integral is

$$\iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y.$$

How do we finish this integral? For the sake of review, let's perform the uv-substitution given by $x = \sqrt{1-u}\cos v$ and $y = \sqrt{1-u}\sin v$, so that $x^2 + y^2$ will simplify to $1 - u$. The constraint $0 \le x^2 + y^2 \le 1$ becomes $0 \le u \le 1$, and to avoid doubling up on points in D, we must take $0 \le v \le 2\pi$. The Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} -\frac{\cos v}{2\sqrt{1-u}} & -\frac{\sin v}{2\sqrt{1-u}} \\ -\sqrt{1-u}\sin v & \sqrt{1-u}\cos v \end{bmatrix} = -\frac{1}{2}\cos^2 v - \frac{1}{2}\sin^2 v = -\frac{1}{2}.$$

So we get the integral

$$\int_{v=0}^{2\pi}\int_{u=0}^{1} \frac{1}{1-(1-u)}\cdot\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v = 2\pi\int_{u=0}^{1}\frac{1}{2\sqrt{u}}\,\mathrm{d}u = 2\pi\sqrt{u}\Big|_{u=0}^{1} = 2\pi.$$

Okay, but let's go back to the part where we still had to solve for z. What's up with that?

The answer is that the formula in Proposition 3.1 does not, actually, let us find the surface area of an implicit surface given by $f(x, y, z) = 0$ without ever solving for z. That is impossible. Unfortunately.

The advantage is that we will at least not have to find the derivatives $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ of the function h giving us $z = h(x, y)$. That's something! Inverse functions are prone to having annoying derivatives, and it's nice to be able to avoid those.

It's also true that sometimes, large parts of the expression $\frac{\|\nabla f\|}{|f\cdot\mathbf{k}}$ simplify before we have to solve for z. In our example, this happened for the numerator, where we could simply replace $x^2+y^2+z^2$ by 1.

We can generalize Proposition 3.1; there's nothing special about the variable z. A more general statement:

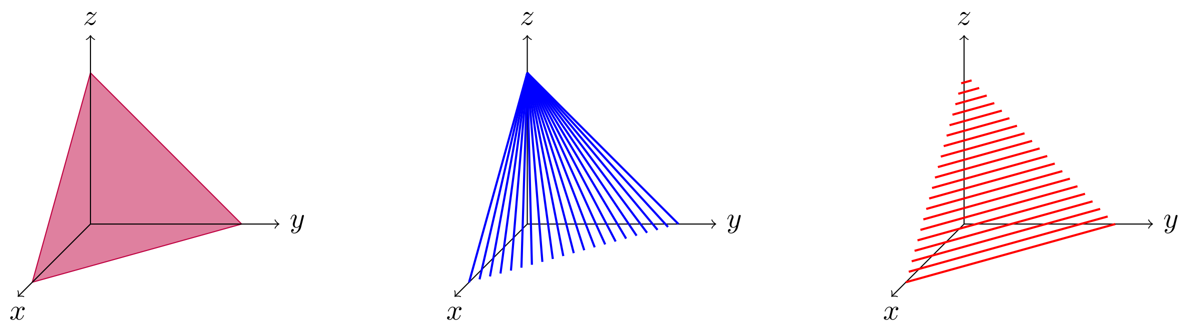
Proposition 3.1. *Suppose that a surface S lies on the graph of the equation $f(x,y,z)=0$, and can be projected onto the xy-, xz-, or the yz-plane to give a region D such that every point on D is the projection of exactly one point on S. Let $\mathbf{p}$ (either $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$) be a unit vector orthogonal to D.*

Then the area of S can be found using the integral

$$\iint_D \frac{\|\nabla f\|}{|\nabla f\cdot\mathbf{p}|}\,\mathrm{d}A.$$

4 Area of a 3D triangle

Let's do a simple example in increasingly complicated ways. What is the area of a triangle with corners $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ shown in Figure 1a?



(a) Just the triangle (b) Parameterization 1 (c) Parameterization 2

Figure 1: The triangle with corners at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

We *could* just use the formula $A=\frac{1}{2}bh$ for the area of a triangle. Let the line segment from $(1,0,0)$ to $(0,1,0)$ be the base; this has length $\sqrt{2}$. Then the height is going to be the length of a line segment from $(0,0,1)$ to the base that is perpendicular to the base; by symmetry, we can take the line segment from $(0,0,1)$ to $(\frac{1}{2},\frac{1}{2},0)$. This has length $\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2+1^2}=\sqrt{\frac{3}{2}}$. Using the formula, we get an area of

$$\frac{1}{2}\cdot\sqrt{2}\cdot\sqrt{\frac{3}{2}}=\frac{1}{2}\sqrt{3}.$$

We could also find a wacky-domain parameterization and take an integral. The three corners of the triangle lie in the plane $x+y+z=1$, and the constraints on x and y are that $x\ge 0$, $y\ge 0$, and $x+y\le 1$. If we take $f(x,y,z)=x+y+z-1$, then $\nabla f=\mathbf{i}+\mathbf{j}+\mathbf{k}$, so $\|\nabla f\|=\sqrt{3}$ and $|\nabla f\cdot\mathbf{k}|=1$. We integrate and get

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\sqrt{3}\,\mathrm{d}y\,\mathrm{d}x=\int_{x=0}^{1}\sqrt{3}(1-x)\,\mathrm{d}x=\sqrt{3}x-\frac{1}{2}\sqrt{3}x^2\Big|_{x=0}^{1}=\frac{1}{2}\sqrt{3}.$$

It is much more effort, but for the sake of practicing parameterizations, we can find some rectangular-domain parameterizations of the triangle. A good way to think about these is to try to describe a family of line segments that "sweep out" the triangle.

Figure 1b shows one such family of line segments. Each blue line segment has one endpoint fixed at $(0, 0, 1)$, and the other endpoint at a point $(x, y, 0)$ moving from $(1, 0, 0)$ to $(0, 1, 0)$. Let's call the top endpoint $\mathbf{a}$ and the bottom endpoint $\mathbf{b}$.

First of all, while $\mathbf{a}$ is a fixed point $(0, 0, 1)$, $\mathbf{b}$ is a variable point; it is a function $\mathbf{b}(u)$ of some parameter u. We know what $\mathbf{b}$ should be doing: it is moving along a line segment from $(1, 0, 0)$ to $(0, 1, 0)$. To describe this segment, we can take $\mathbf{b}(u) = (1 - u, u, 0)$, where $u \in [0, 1]$.

The segments that sweep out our surface are segments from $\mathbf{a}$ to $\mathbf{b}(u)$, which we parameterize by a second parameter, v. Again, because we understand parameterizations of line segments, we know how to do this: this will be given by $\mathbf{r}(u, v) = (1-v)\mathbf{a} + v\mathbf{b}(u)$, where $v \in [0, 1]$. Putting the pieces together, we get

$$\mathbf{r}(u, v) = ((1 - u)v, uv, 1 - v), \qquad (u, v) \in [0, 1] \times [0, 1]. \tag{1}$$

If we want to use this to find the area of the triangle, we will have $\frac{\partial \mathbf{r}}{\partial u} = -v\,\mathbf{i} + v\,\mathbf{j}$ and $\frac{\partial \mathbf{r}}{\partial v} = (1 - u)\,\mathbf{i} + u\,\mathbf{j} - \mathbf{k}$, for a cross product of

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v & v & 0 \\ 1-u & u & -1 \end{bmatrix} = -v\,\mathbf{i} - v\,\mathbf{j} - v\,\mathbf{k}.$$

The norm of this cross product is $\sqrt{3}|v|$. (Just $\sqrt{3}v$ is fine, since $v \in [0, 1]$.) So we integrate

$$\int_{u=0}^{1} \int_{v=0}^{1} \sqrt{3}v \,\mathrm{d}v \,\mathrm{d}u = \int_{v=0}^{1} \sqrt{3}v \,\mathrm{d}v = \frac{\sqrt{3}}{2} v^2 \bigg|_{v=0}^{1} = \frac{1}{2}\sqrt{3}.$$

Let's finish the lecture off with another parameterization, this time thinking of the triangle in the way drawn in Figure 1c. Here, each red line segment connects a point $\mathbf{a}(u)$ and a point $\mathbf{b}(u)$, where

- $\mathbf{a}(u)$, a point moving from $(1, 0, 0)$ to $(0, 0, 1)$, should be given by $\mathbf{a}(u) = (1 - u, 0, u)$, where $u \in [0, 1]$.
- $\mathbf{b}(u)$, a point moving from $(0, 1, 0)$ to $(0, 0, 1)$, should be given by $\mathbf{b}(u) = (0, 1 - u, u)$, where $u \in [0, 1]$.

So each red line segment is given by $\mathbf{r}(u, v) = (1 - v)\mathbf{a}(u) + v\mathbf{b}(u)$, where $v \in [0, 1]$; letting both u and v vary, we get the surface parameterization

$$\mathbf{r}(u, v) = ((1 - u)(1 - v), (1 - u)v, u), \qquad (u, v) \in [0, 1] \times [0, 1]. \tag{2}$$

We don't need to do another integral. Parameterization (2) is equivalent to parameterization (1): if we replace v by $1 - u$ and u by v in (1), we get (2).