Lecture 19: Scalar surface integrals

October 23, 2023 *Kennesaw State University*

1 Defining scalar surface integrals

So far, we've seen how to write an integral for the area of a surface in $\mathbb{R}^3$. Today, we will generalize this to the integral of an arbitrary scalar function over a surface. I am emphasizing that this is a *scalar* surface integral to distinguish it from the *vector* surface integral we will look at in the next lecture.

We define the scalar surface integral by analogy. For scalar line integrals, we generalize the arc length integral

$$\int_C \mathrm{d}s = \int_{t=a}^{b} \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t$$

to the scalar line integral

$$\int_C f \, \mathrm{d}s = \int_{t=a}^{b} f(\mathbf{r}(t)) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t.$$

(Actually, in class, we did this in the other order, but whatever.)

We can generalize the surface area integral to the scalar surface integral in the same way. Let S be a surface in $\mathbb{R}^3$ with parameterization $\mathbf{r}\colon [a,b] \times [c,d] \to \mathbb{R}^3$, and let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a "scalar field" (or, in other words, a real-valued function on $\mathbb{R}^3$). Then we define the integral of f over S to be

$$\iint_S f \, \mathrm{d}S := \int_{u=a}^{b} \int_{v=c}^{d} f(\mathbf{r}(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \mathrm{d}v \, \mathrm{d}u.$$

Just as we wrote $\mathrm{d}s$ to indicate a scalar line integral, we'll write $\mathrm{d}S$ to indicate a scalar surface integral. (Your textbook uses $\mathrm{d}\sigma$; "σ" is the Greek equivalent of "s".) We will call $\mathrm{d}S$ the **differential element of the surface area of** S.[2]

Our definition of the scalar surface integrals can be boiled down to its essential elements if we state it in the following way:

Theorem 1.1. *If S is parameterized by a function $\mathbf{r}$ of u and v, then the differential of the surface area of S is given by*

$$\mathrm{d}S = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \mathrm{d}u \, \mathrm{d}v.$$

I call it a "Theorem" because in the previous lecture, we essentially gave a proof of this claim. The intuition here is the same as it was then: $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\| \, \mathrm{d}u \, \mathrm{d}v$ represents the area of a tiny cell of the

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

[2] This is a very long name, and in practice you can call $\mathrm{d}S$ anything you like, as long as you include the words "differential" or "element" or both, and also the words "surface" or "area" or both.

surface. The integral of f over that tiny cell is approximately just the product of $f(\mathbf{r}(u,v))$ and its area, because f is approximately constant on that tiny cell.

As a special case, when $f(x,y,z) = 1$ for all $(x,y,z) \in \mathbb{R}^3$, so we're just integrating 1 over the entire surface, we get back the surface area integral we studied in the past two lectures.

In the "space dust" interpretation of the scalar field f, where $f(x,y,z)$ tells us how much "space dust" there is at a point (x,y,z), the surface integral of f over S tells us the total amount of space dust on the surface S. This is not really a practical application, but it can help your intuition. For example, it tells us that changing the parameterization $\mathbf{r}$ will not affect the surface integral, as long as we describe the same surface.

To start off with a simple example, let's take the sloped rectangle with corners at $(0,0,1)$, $(1,0,1)$, $(1,1,0)$, and $(0,1,0)$, shown in Figure 1, and integrate $f(x,y,z) = xyz$ over it.

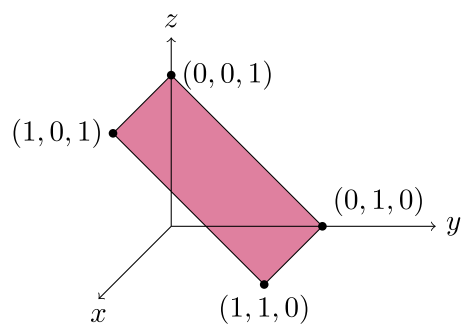

Figure 1: The rectangle with corners at $(0,0,1)$, $(1,0,1)$, $(1,1,0)$, and $(0,1,0)$.

We can parameterize the rectangle by $\mathbf{r}(u,v) = (u, v, 1-v)$ where $(u,v) \in [0,1] \times [0,1]$. One way to see this is that we're parameterizing in terms of the (x,y)-coordinates. Another way it so think of this as a family of line segments from $(u,0,1)$ to $(u,1,0)$, where u also varies from 0 to 1.

If we compute $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|$, we get $\mathbf{j} + \mathbf{k}$, which makes sense: this is a constant vector perpendicular to the plane of the rectangle. The norm of $\mathbf{j} + \mathbf{k}$ is $\sqrt{2}$, so we conclude that $\mathrm{d}S = \sqrt{2}\,\mathrm{d}u\,\mathrm{d}v$. Now, we just integrate $f(\mathbf{r}(u,v)) = uv(1-v)$ with a factor of $\sqrt{2}$ on it:

$$\int_{u=0}^{1} \int_{v=0}^{1} \sqrt{2}uv(1-v)\,\mathrm{d}v\,\mathrm{d}u = \sqrt{2}\left(\int_{u=0}^{1} u\,\mathrm{d}u\right)\left(\int_{v=0}^{1} (v - v^2)\,\mathrm{d}v\right).$$

The integral with respect to u gives $\frac{1}{2}$, the integral with respect to v gives $\frac{1}{6}$, so in the end, we get $\sqrt{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{\sqrt{2}}{12}$.

2 Physical applications

We used scalar line integrals to find the mass of a thin wire in the shape of the curve C, given a function $\delta(x,y,z)$ giving us the "mass per unit length" along the wire at a point (x,y,z). Alternatively, if $\delta(x,y,z)$ measures the cross-sectional area of the wire, then the scalar line integral would give the volume of the wire.

Similarly, suppose we have an object in the shape of a thin shell. Then we can compute its mass and volume by an appropriate surface integral. If $\delta(x, y, z)$ measures the *thickness* of the shell at a point (x, y, z), then integrating $\delta(x, y, z)$ over the surface will give the volume of the object. If $\delta(x, y, z)$ measures the "mass per unit area" of the surface at a point (x, y, z), then integrating $\delta(x, y, z)$ over the surface will give the mass of the object.

This is only an approximation, valid when the thickness of the shell is negligible relative to its other dimensions. Let's begin by looking at a very easy example where we can *see* how good of an approximation it is. What is the volume of a sphere of radius 1 with a *constant* thickness δ?

First, let's set up a scalar surface integral, and then not do it. We saw in a previous lecture that if we take the parameterization

$$\rho(\phi, \theta) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \qquad (\phi, \theta) \in [0, \pi] \times [0, 2\pi]$$

then $\|\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\| \, \mathrm{d}S$ will simplify to $\sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta$. (For a sphere of radius a, it will simplify to $a^2 \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta$.) So we can take the integral

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \delta \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

Why aren't we doing this integral? Well, we can factor out δ (it's a constant), and be left with the surface area integral. We did this surface area integral in a previous lecture and already found the formula for the surface area of a sphere. When the radius of the sphere is 1, the area is 4π, so the volume of our shell will be $4\pi\delta$.

That was the approximation; now let's compute the real volume. We can do this with the spherical integral

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=1-\delta/2}^{1+\delta/2} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

(Here, I'm assuming "thickness δ" means that the shell extends by a distance of $\delta/2$ inward $\delta/2$ outward from the sphere of radius 1.)

We can factor out the part of this integral that depends on ρ, getting

$$\left(\int_{\rho=1-\delta/2}^{1+\delta/2} \rho^2 \, \mathrm{d}\rho\right) \left(\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta\right).$$

As before, the integral with respect to ϕ and θ simplifies to 4π. The integral with respect to ρ simplifies to

$$\left.\frac{\rho^3}{3}\right|_{\rho=1-\delta/2}^{1+\delta/2} = \frac{1}{3}\left(1 + \frac{\delta}{2}\right)^3 - \frac{1}{3}\left(1 - \frac{\delta}{2}\right)^3 = \delta + \frac{1}{12}\delta^3.$$

Multiplying by 4π, we conclude that the volume of the spherical shell is *actually* $4\pi\delta + \frac{1}{3}\pi\delta^3$.

The approximation of $4\pi\delta$ is not that bad, then. Let's imagine that we're computing the volume of the earth's atmosphere. Actually, the earth's atmosphere varies in height considerably, but let's approximately say it's 12 km everywhere. Well, the radius of the Earth is $R_\oplus \approx 6\,371$ km on average, so the thickness δ is about $0.00188\, R_\oplus$. Our calculation with the surface integral would

$4\pi\delta \approx 0.0237\, R_\oplus^3$ for the volume of the Earth's atmosphere, and the error that we incurred by not taking the volume integral properly was $\frac{1}{3}\pi\delta^3 \approx 6.99 \cdot 10^{-7}\, R_\oplus^3$.

(On the other hand, if our thickness δ were 1, equal to the radius of the sphere itself, then one method would give us a volume of 4π and the other would give us a volume of $4\pi + \frac{1}{3}\pi$, which is pretty bad.)

3 Variations on the integral

All of the usual tricks we did with surface area in the previous lecture can also be used here. For example, suppose we have a surface given by $z = h(x, y)$ above a region R in the xy-plane. Last time, we found that the surface area of such a region is given by

$$\iint_R \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}\, \mathrm{d}A.$$

Let's restate this claim as follows:

Theorem 3.1. *If S is a surface given by the equation $z = h(x, y)$ above a region R in the xy-plane, then the differential of the surface area of S is given by*

$$\mathrm{d}S = \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}\, \mathrm{d}A.$$

(Here, $\mathrm{d}A$ could also be taken as a differential element of a surface area. But the surface area of R is just the area of R, since R is a flat region in the xy-plane, so this is a much nicer differential element; when the time comes, we'll just replace it by $\mathrm{d}x\, \mathrm{d}y$.)

As a result, if we want to integrate a function f over this surface, we take the integral

$$\iint_R f(x, y, h(x, y)) \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}\, \mathrm{d}A.$$

Let's do an example. Say that our surface lies on the graph of $z = y^2$ and above the triangle $\{(x, y) \in \mathbb{R}^2 : 0 \le x \le y \le 1\}$ in the xy-plane, and we want to integrate $f(x, y, z) = x + y + z$ over this surface. Then we get

$$\iint_S xyz\, \mathrm{d}S = \int_{x=0}^1 \int_{y=x}^1 (x + y + y^2)\sqrt{0 + (2y)^2 + 1}\, \mathrm{d}y\, \mathrm{d}x.$$

Here, $\int_{x=0}^1 \int_{y=x}^1$ is our description of the "shadow" of our surface in the xy-plane; $(x + y + y^2)$ is the function $f(x, y, z) = xyz$ evaluated at a point where $z = y^2$; finally, $\sqrt{0 + (2y)^2 + 1}\, \mathrm{d}y\, \mathrm{d}x$ is the area element for our surface integral.

Mathematica tells me that the result of taking this integral is $\frac{1}{120} + \frac{121}{192}\sqrt{5} - \frac{3}{128} \operatorname{arcsinh} 2$.

Similarly, suppose a surface S lies on the graph of $f(x, y, z) = 0$, and can be projected onto the xy-, xz-, or yz-plane to give a region D such that every point on D is the projection of exactly one point on S. Last time, we found that the surface area of S is exactly

$$\iint_D \frac{\|\boldsymbol{\nabla} f\|}{|\boldsymbol{\nabla} f \cdot \mathbf{p}|}\, \mathrm{d}A$$

where $\mathbf{p}$ is either $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$, whichever is orthogonal to the plane of D. Now, let's state this in terms of the differential element:

Theorem 3.2. *Let S be a surface that lies on the graph of the implicit equation $f(x, y, z) = 0$, and can be projected onto a plane with normal vector $\mathbf{p}$ such that every point of S projects to exactly one point on the plane. Then the differential of the surface area of S is given by*

$$\mathrm{d}S = \frac{\|\boldsymbol{\nabla} f\|}{|\boldsymbol{\nabla} f \cdot \mathbf{p}|}\, \mathrm{d}A$$

where $\mathrm{d}A$ is the differential of the surface area of the plane.[3]

Awkwardly, we are using the letter f in the equation $f(x, y, z) = 0$, so we can no longer use f for the function we want to integrate; as a result, I'll skip the general formula. But the idea is the same: to change the surface area integral to a scalar surface integral, just include the function you want to integrate inside that integral.

For example, let's integrate xyz over half of an upright cylinder: the cylinder $x^2 + y^2 = 1$ with $y \geq 0$ and $0 \leq z \leq 1$. Instead of using cylindrical coordinates, we'll realize that the cylinder has a nice projection onto the xz -plane: the rectangle $D = \{(x, z) : -1 \leq x \leq 1, 0 \leq z \leq 1\}$. So we write

$$\iint_S xyz\, \mathrm{d}S = \int_{x=-1}^{1} \int_{z=0}^{1} xyz \frac{\|\boldsymbol{\nabla} f\|}{|\boldsymbol{\nabla} f \cdot \mathbf{p}|}\, \mathrm{d}z\, \mathrm{d}x.$$

Since $f(x, y, z) = x^2 + y^2 - 1$, we get $\boldsymbol{\nabla} f = 2x\,\mathbf{i} + 2y\,\mathbf{j}$, with norm $\|\boldsymbol{\nabla} f\| = 2\sqrt{x^2 + y^2}$. The vector $\mathbf{p}$ is $\mathbf{j}$: the vector perpendicular to the xz-plane; $\boldsymbol{\nabla} f \cdot \mathbf{j}$ gives us $2y$. So we integrate

$$\int_{x=-1}^{1} \int_{z=0}^{1} xyz \frac{2\sqrt{x^2 + y^2}}{2y}\, \mathrm{d}z\, \mathrm{d}x.$$

The integrand simplifies before we integrate it: $\sqrt{x^2 + y^2}$ is identically 1 on our surface, and the y cancels with the y in xyz. So we are just integrating xz over the rectangle D. We get 0, because we are integrating an odd function of x over the interval $-1 \leq x \leq 1$.

[3] Sneakily, this also allows us to project S onto a plane that is not the xy-, xz-, or yz-plane. But we rarely have a reason to do this, and I won't throw such things at you in this class; you have enough on your plate.