Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 2: Spherical Coordinates

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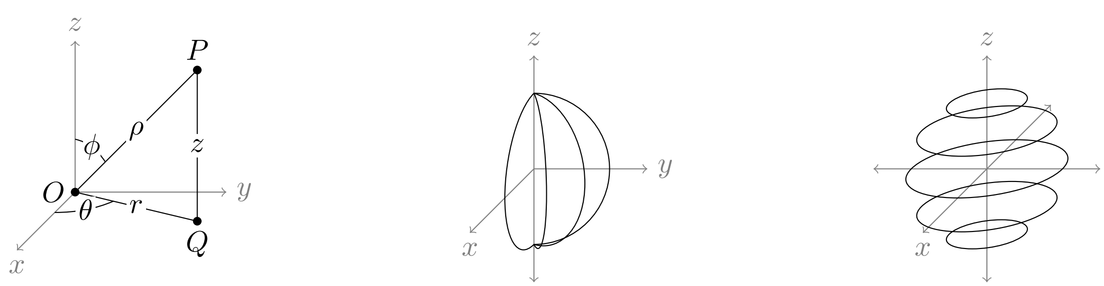
1 Points in spherical coordinates

Spherical coordinates are a second generalization of polar coordinates to three dimensions. The difference is that cylindrical coordinates kept the old polar variables, and added a new rectangular dimension. Spherical coordinates, on the other hand, keep the *philosophy* of polar coordinates: they specify a point in three dimensions by giving a distance from the origin, and a direction. The direction now takes two angular coordinates to fully specify.

The first way to understand spherical coordinates is to start from our cylindrical picture of a point P and its "shadow" Q; see Figure 1a. We will keep the θ coordinate, since it tells part of the story for the direction to P, but the other cylindrical coordinates are only there for comparison.

We will use ρ for the spherical radius: this is the distance from the origin O to the point P.

Finally, once we've specified θ and ρ, our only degree of freedom is to "swing" the fixed-length segment OP up and down. We let our third coordinate ϕ be the angle it makes with the positive z-axis. The range of ϕ is from 0 to π. If $\phi = 0$, then P is directly above the origin; if $\phi = \frac{\pi}{2}$, then P is in the xy-plane; if $\phi = \pi$, then P is directly below the origin.



(a) The spherical coordinates of point P

(b) Several curves of constant θ on the sphere $\rho = 1$

(c) Several curves of constant ϕ on the sphere $\rho = 1$

Figure 1: Two ways to understand spherical coordinates

Another way to gain intuition for spherical coordinates is to visualize them on a sphere. Let's begin by keeping ρ constant: for concreteness, say $\rho = 1$. This puts us on a sphere of radius 1 centered at the origin.

If we pick a constant θ, and let ϕ vary from 0 to π, then we trace out a half-circle of radius 1 (Figure 1b). If we imagine that the sphere of radius 1 is the Earth, then θ gives us the longitude (the east-west position), so the lines of constant θ are lines from the North Pole to the South Pole.

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

If we pick a constant ϕ, and let θ vary from 0 to 2π, then we trace out a circle of varying radius with center on the z-axis (Figure 1c). If we imagine that the sphere of radius 1 is the Earth, then ϕ gives us the latitude (the north-south position), so the lines of constant ϕ are circles that loop around the Earth in a plane parallel to the equator.

Be careful: the mathematical convention is different from the geographical one here! On Earth, the latitude ranges from 90° N to 90° S, which we could think of as $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, where 0 is the equator. Our ϕ coordinate adds $\frac{\pi}{2}$ to this coordinate: 0 is the North Pole, $\frac{\pi}{2}$ is the equator, and π is the South Pole.

Returning to Figure 1a, we can try to relate spherical coordinates to rectangular ones by going through cylindrical coordinates. Take a look at triangle $\triangle OPQ$; it is a right triangle with hypotenuse ρ whose sides are r and z. The Pythagorean theorem says that $\rho^2 = r^2 + z^2$; it is also true that $r^2 = x^2 + y^2$, so we get $\rho^2 = x^2 + y^2 + z^2$.

In this right triangle, $\angle O$ is $\frac{\pi}{2} - \phi$ (at least the way we've drawn P: with positive z-coordinate). Therefore $\frac{z}{\rho} = \sin\angle O$ and $\frac{r}{\rho} = \cos\angle O$, which tells us that $z = \rho\sin(\frac{\pi}{2} - \phi) = \rho\cos\phi$, while $r = \rho\cos(\frac{\pi}{2} - \phi) = \rho\sin\phi$.

We already know that $x = r\cos\theta$ and $y = r\sin\theta$, and putting these together gives us the formulas relating spherical coordinates to rectangular coordinates:

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

These are not always easy to remember. If you're worried that you've messed up, try some special cases: for example, make sure that $\phi = 0$ gives a point on the positive z-axis, and $\phi = \frac{\pi}{2}$ and $\theta = 0$ gives you a point on the positive x-axis.

2 Integrating in spherical coordinates

To integrate in spherical coordinates, we must replace $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ by some volume element $\underline{\ ???\ }\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$. Let's figure out what goes in the blank.

We first chop up space into nested shells of thickness $\Delta\rho$. Taking one such shell with radius ρ, we futher chop up its surface by curves of constant θ (spaced $\Delta\theta$ radians apart) and curves of constant ϕ (spaced $\Delta\phi$ radians apart). You have already seen this picture: it is a globe, with parallels and meridians dividing its surface into cells.

Let's start with the height of one of these cells. The overall distance from pole to pole is $\pi\rho$: halfway around a circle of radius ρ. But we only go $\frac{\Delta\phi}{\pi}$ of that distance, so the height is $\rho\Delta\phi$.

The width is trickier: you can see that the width of a cell varies depending on where we are on the sphere. That's because it actually depends on r (the cylindrical radius: distance from the z-axis) rather than ρ (the spherical radius: distance from the origin). In cylindrical coordinates, the width of a cell would be $r\Delta\theta$: a quantity we already saw in the previous lecture. Since $r = \rho\sin\phi$, the width of a cell for us is $\rho\sin\phi\Delta\theta$.

When $\Delta\rho$, $\Delta\theta$, and $\Delta\phi$ is small, a cell on the spherical shell has volume approximately $\Delta\rho \times \rho\Delta\phi \times \rho \sin\phi \Delta\theta$. Therefore the quantity that replaces $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in our integrals must be $\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

A quick application you should feel free to try on your own: take an integral as θ goes from 0 to 2π, ϕ goes from 0 to π, and ρ goes from 0 to a constant R. This should give you the volume of a sphere of radius R, for which the formula is $\frac{4}{3}\pi R^3$.

3 A spherical integral with center of mass

Let's work through an example.

First, a question: what region does the inequality $0 \le \phi \le \frac{\pi}{6}$ describe?

Since there is no constraint on θ, this region will be rotationally symmetric around the z-axis. Since there is no constraint on ρ, it will extend infinitely far along straight rays out from the origin. Finally, the constraint on ϕ tells us that we're limited to rays that point in directions close to upward. (Specifically, $\phi = \frac{\pi}{6}$ corresponds to $60°N$ on Earth; this is close to the latitude of Anchorage, Alaska.)

Therefore this region is an infinite solid cone, opening upward from the origin (that is, in "ice cream cone" orientation). This is a region we can also easily describe in cylindrical coordinates, via the relationship $r = z\tan\phi$. Here, the same cone is described by $z \ge r\sqrt{3}$.

What if we want to cut off the cone at height $z = 1$? This would be easy in cylindrical coordinates. In spherical coordinates, we should rewrite $0 \le z \le 1$ as $0 \le \rho\cos\phi \le 1$. In particular, if we wanted to integrate over this finite cone, we could put an upper bound of $\frac{1}{\cos\phi}$ on ρ, writing an integral of the form

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/6} \int_{\rho=0}^{1/\cos\phi} \underline{\qquad}\ \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

We wouldn't make our lives complicated like this without a reason, so let's give ourselves a reason.

Last time, we talked about the centroid of a region. A more concrete version of the centroid is the center of mass of a physical object. The center of mass is basically the point where the object will balance in any direction; the point where, if you apply force, it will just go in that direction and not spin.

If an object has uniform density, the centroid and center of mass are the same. But what if the density is not uniform?

If an object has density $\delta(x, y, z)$ at point (x, y, z), then we can find its mass by integrating $\delta(x, y, z)$ over the region. Similarly, if we want to find the center of mass, we should include the density in all of our integrals. The coordinates $(\overline{x}, \overline{y}, \overline{z})$ of the center of mass of a region R with density δ are given by

$$\overline{x} = \frac{\iiint_R x\,\delta\,\mathrm{d}V}{\iiint_R \delta\,\mathrm{d}V} \qquad \overline{y} = \frac{\iiint_R y\,\delta\,\mathrm{d}V}{\iiint_R \delta\,\mathrm{d}V} \qquad \overline{z} = \frac{\iiint_R z\,\delta\,\mathrm{d}V}{\iiint_R \delta\,\mathrm{d}V}$$

One reason that we might prefer to describe our cone in spherical coordinates is if the density δ is best described in spherical coordinates. For example, what if the cone increases in density as we go out from the origin: what if $\delta(\rho, \phi, \theta) = \rho$?

In this case, our mass integral would be:

$$\begin{aligned}
\text{Mass} &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/6} \int_{\rho=0}^{1/\cos\phi} \rho^3 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= 2\pi \int_{\phi=0}^{\pi/6} \int_{\rho=0}^{1/\cos\phi} \rho^3 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \\
&= 2\pi \int_{\phi=0}^{\pi/6} \left. \frac{\rho^4 \sin\phi}{4} \right|_{\rho=0}^{1/\cos\phi} \mathrm{d}\phi \\
&= 2\pi \int_{\phi=0}^{\pi/6} \frac{\sin\phi}{4\cos^4\phi} \, \mathrm{d}\phi \\
&= 2\pi \int_{u=1}^{\sqrt{3}/2} \left(-\frac{1}{4u^4}\right) \mathrm{d}u \qquad (u = \cos\phi,\ \mathrm{d}u = -\sin\phi \, \mathrm{d}\phi) \\
&= 2\pi \cdot \left. \frac{1}{12u^3} \right|_{u=1}^{\sqrt{3}/2} = \frac{4\pi}{9\sqrt{3}} - \frac{\pi}{6}.
\end{aligned}$$

To find the center of mass, we have to do three more integrals. Fortunately, just as before, two of the integrals can be avoided by use of symmetry: both the region R and its density are symmetric in the x-direction and y-direction, so $\overline{x} = \overline{y} = 0$.

To find $\overline{z}$, we have to include a factor of $z = \rho\cos\phi$ in our integral. This results in

$$\begin{aligned}
\iiint_R z \, \delta \, \mathrm{d}V &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/6} \int_{\rho=0}^{1/\cos\phi} \rho^4 \sin\phi \cos\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= 2\pi \int_{\phi=0}^{\pi/6} \int_{\rho=0}^{1/\cos\phi} \rho^4 \sin\phi \cos\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \\
&= 2\pi \int_{\phi=0}^{\pi/6} \left. \frac{\rho^5}{5} \sin\phi \cos\phi \right|_{\rho=0}^{1/\cos\phi} \mathrm{d}\phi \\
&= 2\pi \int_{\phi=0}^{\pi/6} \frac{\sin\phi}{5\cos^4\phi} \, \mathrm{d}\phi
\end{aligned}$$

at which point we throw our hands up in dismay: half of the steps we took when computing our mass integral were unnecessary!

Indeed, if we integrated $\frac{\sin\phi}{4\cos^4\phi}$ for the mass and got an answer of A, then integrating $\frac{\sin\phi}{5\cos^4\phi}$ will yield an answer of $\frac{4}{5}A$. Dividing one by the other will give us $\overline{z} = \frac{\frac{4}{5}A}{A} = \frac{4}{5}$, no matter what A turns out to be.

In conclusion, the center of mass of our non-uniform ice cream cone is the point $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, \frac{4}{5})$.