Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 20: Vector surface integrals

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1 An overview of vector surface integrals

Last time, we talked about scalar integrals over a surface S; this time, we will talk about vector integrals.

The vector surface integral is a flux integral. In $\mathbb{R}^2$, a flux integral across a curve measures the amount that a vector field $\mathbf{F}$ is crossing that curve. In $\mathbb{R}^3$, a flux integral across a surface measures the amount that a vector field $\mathbf{F}$ is crossing that surface.[2] One way we will write this flux integral is as

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S,$$

and this notation will also lead us to one possible way of calculating the flux integral.

There is another way that we can write the flux integral across a surface: if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, then we can write the flux integral of $\mathbf{F}$ across S as

$$\iint_S M \, \mathrm{d}y \wedge \mathrm{d}z + N \, \mathrm{d}z \wedge \mathrm{d}x + P \, \mathrm{d}x \wedge \mathrm{d}y.$$

One reason that I like this notation is that it correctly reveals the vector surface integral as the simplest and most natural kind of integral to take over a surface—which just so happens to measure flux. (With the other notation, it feels a bit like we picked out flux as an arbitrary thing to measure out of a bucket of some unknown number of interactions between $\mathbf{F}$ and S.) We can use the rules for manipulating wedge products to turn this integral into something we can evaluate.

You will also sometimes see this second expression written without the wedge products, as an integral of $M \, \mathrm{d}y \, \mathrm{d}z + N \, \mathrm{d}z \, \mathrm{d}x + P \, \mathrm{d}x \, \mathrm{d}y$. This is misleading, because no matter how we think of the vector surface integral, it is an *oriented* integral; it depends on the orientation of the surface S.

This means that before we define the vector surface integral properly, we will have to go on a digression: what does it mean for a surface to be oriented?

2 Oriented surfaces

2.1 What we know about orientation so far

Let's begin by summarizing the notions of orientation we've discussed so far this semester.

[1]This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

[2]We could define a flux integral in $\mathbb{R}^n$ for any $n \geq 2$, where we would take an $(n-1)$-dimensional hypersurface, and measure how much a vector field is crossing it.

First: to orient a curve C, pick a positive direction of travel along C. A parameterization of C does the trick: if C is parameterized by $\mathbf{r}(t)$ where $t \in [a, b]$, then the positive direction of travel along C is the direction of travel that corresponds to increasing t from a to b. Locally, $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is a vector tangent to C that points in the positive direction along C.

We have a second notion of orientation for a curve C: a choice of positive direction of *crossing* C. This comes in handy when we're doing a flux integral across C. We've picked a convention for how to make these two notions of orientation of C correspond to each other: the positive direction of *crossing* C is the direction that points from left to right, from the point of view of a particle traveling in the positive direction *along* C. This convention corresponds to a cross product: if $\mathbf{T}$ is a tangent vector in the positive direction along C, then $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is a normal vector in the positive direction crossing C.

Finally, we have a notion of orientation for a region R in the plane. An oriented region R has a positive *direction of rotation* at each point. (The standard choice is to pick the counterclockwise direction as positive.) What's more, if the boundary of R is a closed curve C, then an orientation of R goes hand in hand with an orientation of C: we orient the curve C to travel around R in whichever direction R considers the positive direction of rotation.

2.2 Orienting a surface in three dimensions

Now let S be a surface in $\mathbb{R}^3$. We will have two ways of thinking about the orientation of S: one that generalizes what we did for curves, and one that generalizes what we did for regions in the plane. We will adopt a convention (which ultimately stems from the right-hand rule) for how these two notions of orientation of S correspond to each other.

As for regions in the plane, one notion of orientation for S is a choice of positive direction of rotation on S. Unfortunately, this positive direction cannot be simply thought of as "clockwise" or "counterclockwise", because both of these depend on the perspective of the observer. Imagine going up to a window and drawing a counterclockwise loop on it in marker. From the point of view of someone standing outside the window, the loop you've drawn goes clockwise!

A parameterization $\mathbf{r}(u, v)$ of S tells us a choice of positive direction of rotation on S, though it's a bit roundabout. The vectors $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are both tangent vectors to S. If you take a tangent vector to S and rotate it while keeping it tangent at the same point, you will either go through the directions $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, -\frac{\partial \mathbf{r}}{\partial u}, -\frac{\partial \mathbf{r}}{\partial v}$ in that order, or in the reverse order. The direction of rotation that seems them in the order given above is the positive one.

(From this definition, we see that one quick way to reverse the orientation given by a parameterization is to swap the variables u and v.)

As for curves in the plane, a second notion of orientation for S is a choice of positive direction for crossing S. This is relevant for flux, because flux is all about vectors crossing S. Equivalently, this second notion of orientation for S is all about picking a unit-length normal vector $\mathbf{n}$ at every point of S. (We want to be consistent about this choice: when we move around S, the normal vector $\mathbf{n}$ might of course change, but it should change continuously, and never suddenly flip its sign.)

The convention for going from one notion of orientation to the other is this: if you're looking at S so that the normal vector $\mathbf{n}$ is pointing *at you*, then the positive direction of rotation is *counterclockwise* from your point of view. A parameterization $\mathbf{r}(u, v)$ of S also tells us a choice of positive direction for crossing S, which respects this convention. The rule is: pick a normal vector $\mathbf{n}$ which points in the same direction as $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$. In other words,

$$\mathbf{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|}.$$

(Just as before: if you change $\mathbf{r}(u, v)$ to $\mathbf{r}(v, u)$, then you swap the order of $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ in the cross product, which turns $\mathbf{n}$ to $-\mathbf{n}$, reversing its orientation.)

3 Finally defining the vector surface integral

Earlier, we defined the flux integral of $\mathbf{F}$ across a surface S as

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S.$$

Now, we can make sense of this.

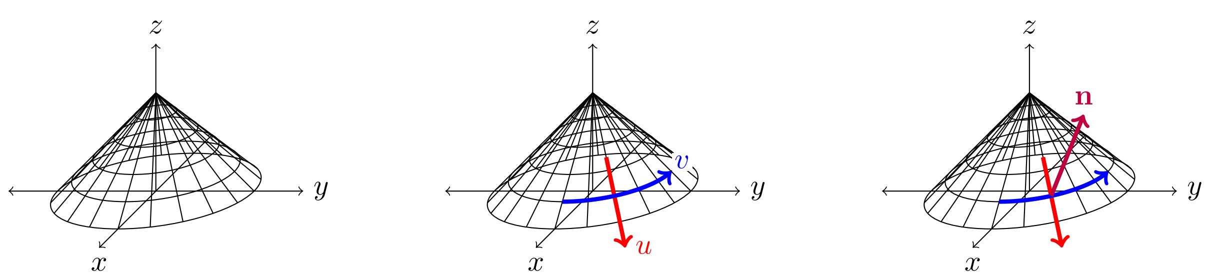
As before, when we've parameterized S by $\mathbf{r}(u, v)$, where $(u, v) \in [a, b] \times [c, d]$, a surface integral replaces $\mathrm{d}S$ by $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\| \, \mathrm{d}u \, \mathrm{d}v$. However, this time, the quantity we're integrating is $\mathbf{F} \cdot \mathbf{n}$, which depends on S as well. As usual, $\mathbf{F}$ will just become $\mathbf{F}(\mathbf{r}(u, v))$: we evaluate $\mathbf{F}$ at a point on the surface. Meanwhile, $\mathbf{n}$ is the unit normal vector we've defined in the previous section: it is $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ divided by $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|$.

We are dividing by $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|$, but then we are also multiplying by it! The two effects cancel out to give a somewhat simpler expression:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int_{u=a}^{b} \int_{v=c}^{d} \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \mathrm{d}v \, \mathrm{d}u.$$

Before we say anything more, let's do an example.

Let S be the lateral boundary of the cone whose base is the unit circle in the xy-plane, and whose vertex is at point $(0, 0, 1)$ (shown in Figure 1a). What is the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ *outward and upward* through S?



(a) The surface S (b) u and v directions (c) Normal vector $\mathbf{n}$

Figure 1: A conical surface and its orientation

We can use cylindrical coordinates to parameterize S. In cylindrical coordinates, the equation of the cone is $z = 1 - r$, where $0 \leq r \leq 1$. If we take $r = u$ and $\theta = v$, then $z = 1 - u$, and we get the parameterization

$$\mathbf{r}(u, v) = (u \cos v, u \sin v, 1 - u), \qquad (u, v) \in [0, 1] \times [0, 2\pi].$$

But does this parameterization have the right orientation we want?

One way to check is to think about the directions of increasing u and increasing v on the cone (shown in Figure 1b). When u increases, we move from the tip of the cone to the rim. When v increases, we move around the cone, in the direction that's counterclockwise viewed from above.

The orientation we give S of the positive-direction-of-rotation flavor attaches a rotation from the increasing-u direction to the increasing-v direction at every point. That rotation looks like a counterclockwise rotation when viewed from outside the cone (the perspective we have in Figure 1b). So the positive direction of crossing S should be the one that assigns the normal vector out at the viewer in that picture (as in Figure 1c).

There is also an algebraic approach. If we compute $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$, which we'll need anyway, we get

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & -1 \\ -u \sin v & u \cos v & 0 \end{bmatrix} = u \cos v \, \mathbf{i} + u \sin v \, \mathbf{j} + u \, \mathbf{k}.$$

We can try it out at a test point like $\mathbf{r}(\frac{2}{3}, 0) = (\frac{2}{3}, 0, \frac{1}{3})$, and get the vector $\frac{2}{3} \mathbf{i} + \frac{2}{3} \mathbf{k}$. This vector sure looks like it points up and out of the cone!

Now that we're certain of our orientation, we can start setting up the integral. We have $\mathbf{F}(\mathbf{r}(u, v)) = u \cos v \, \mathbf{i} + u \sin v \, \mathbf{j} + u(1 - u)$. Taking the dot product, we get

$$\mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) = u^2 \cos^2 v + u^2 \sin^2 v + u(1 - u) = u^2 + u(1 - u) = u.$$

Now we can integrate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \int_{u=0}^{1} \int_{v=0}^{2\pi} u \, \mathrm{d}v \, \mathrm{d}u = 2\pi \int_{u=0}^{1} u \, \mathrm{d}u = \left. \frac{2\pi u^2}{2} \right|_{u=0}^{1} = \pi.$$

Another check that we have the right orientation can sometimes be done at this stage: if we think about how the vector $\mathbf{F}$ points, we do expect it to have a positive flux across S, not a negative flux. (This is less reliable when we expect $\mathbf{F} \cdot \mathbf{n}$ to be positive at some points and negative at others, and we aren't sure which sign dominates.)

If you like, there is another way to determine what the integrand is. Recall that for vectors $\mathbf{a} = a_1 \, \mathbf{i} + a_2 \, \mathbf{j} + a_3 \, \mathbf{k}$, $\mathbf{b} = b_1 \, \mathbf{i} + b_2 \, \mathbf{j} + b_3 \, \mathbf{k}$, and $\mathbf{c} = c_1 \, \mathbf{i} + c_2 \, \mathbf{j} + c_3 \, \mathbf{k}$, we have

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Therefore in order to compute $\mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)$, we can take the determinant of a 3×3 matrix whose rows are $\mathbf{F}(\mathbf{r}(u,v))$, $\frac{\partial \mathbf{r}}{\partial u}$, and $\frac{\partial \mathbf{r}}{\partial v}$. In our case, we'd get

$$\det \begin{bmatrix} u\cos v & u\sin v & 1-u \\ \cos v & \sin v & -1 \\ -u\sin v & u\cos v & 0 \end{bmatrix}.$$

Try it out—it simplifies to u!

Finally, let's look at what happens with the other approach: what if we think of the surface integral as

$$\iint_S M\,\mathrm{d}y \wedge \mathrm{d}z + N\,\mathrm{d}z \wedge \mathrm{d}x + P\,\mathrm{d}x \wedge \mathrm{d}y?$$

Here, use $x = u\cos v$, $y = u\sin v$, and $z = 1-u$ to write

$$\begin{aligned} \mathrm{d}x &= \cos v\,\mathrm{d}u - u\sin v\,\mathrm{d}v, \\ \mathrm{d}y &= \sin v\,\mathrm{d}u + u\cos v\,\mathrm{d}v, \\ \mathrm{d}z &= -\mathrm{d}u. \end{aligned}$$

Now, the three parts of the integral are

$$\begin{aligned} M\,\mathrm{d}y \wedge \mathrm{d}z &= u\cos v\,(\sin v\,\mathrm{d}u + u\cos v\,\mathrm{d}v) \wedge (-\mathrm{d}u) = u^2\cos^2 v\,\mathrm{d}u \wedge \mathrm{d}v, \\ N\,\mathrm{d}z \wedge \mathrm{d}x &= u\sin v\,(-\mathrm{d}u) \wedge (\cos v\,\mathrm{d}u - u\sin v\,\mathrm{d}v) = u^2\sin^2 v\,\mathrm{d}u \wedge \mathrm{d}v, \\ P\,\mathrm{d}x \wedge \mathrm{d}y &= (1-u)\,(\cos v\,\mathrm{d}u - u\sin v\,\mathrm{d}v) \wedge (\sin v\,\mathrm{d}u + u\cos v\,\mathrm{d}v) \\ &= (1-u)(u\cos^2 v + u\sin^2 v)\,\mathrm{d}u \wedge \mathrm{d}v. \end{aligned}$$

Together, they add up to $u\,\mathrm{d}u \wedge \mathrm{d}v$.

The conclusion is that the surface integral simplifies to the oriented integral

$$\iint_D u\,\mathrm{d}u \wedge \mathrm{d}v$$

where D is the oriented domain of our parameterization. One way to think of this oriented integral is that $\mathrm{d}u \wedge \mathrm{d}v$ is positive if the u-to-v direction of rotation on our surface is positive, which goes back to Figure 1b again. The answer in the case of the cone is: yes, this direction is positive. Therefore we are free to take the integral of $u\,\mathrm{d}u \wedge \mathrm{d}v$ and turn it into the unoriented integral of $u\,\mathrm{d}u\,\mathrm{d}v$.