Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 24: Stokes' theorem in action

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1 An example of Stokes' theorem

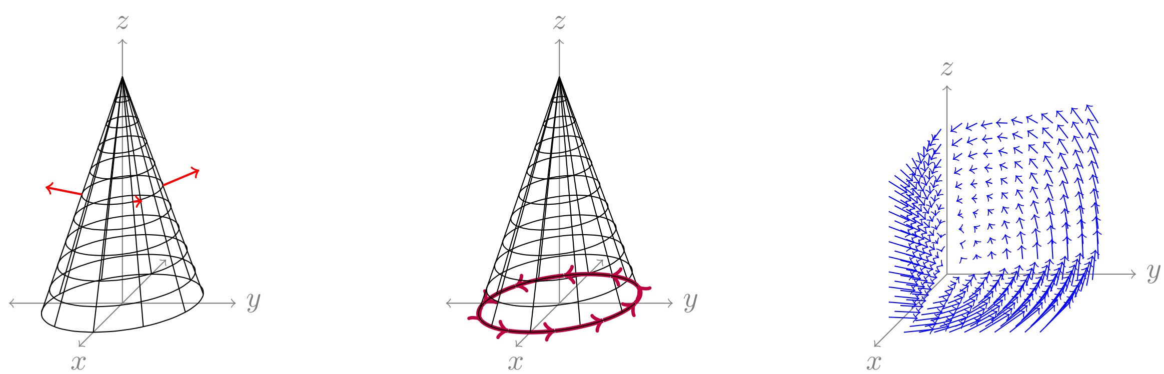
Let's work through an example of both sides of the equation

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S$$

that Stokes' theorem tells us.

Suppose that our surface S is the cone parameterized by $\mathbf{r}(u, v) = (u \cos v, u \sin v, 3 - 3u)$, where $(u, v) \in [0, 1] \times [0, 2\pi]$. If we compute $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$, we get $3u \cos v \, \mathbf{i} + 3u \sin v \, \mathbf{j} + u \, \mathbf{k}$; this tells us that the normal vectors point outward, as in Figure 1a.

The only nontrivial boundary of S is the curve C parameterized by $\mathbf{r}(1, t)$, where $t \in [0, 2\pi]$; all other boundaries cancel out. This is $\mathbf{r}(1, t) = (\cos t, \sin t, 0)$: the unit circle, counterclockwise when seen from above, shown in Figure 1b.



(a) The surface S (b) The boundary of S (c) The vector field $\mathbf{F}$

Figure 1: The three important features of our example

Finally, let's take the vector field $\mathbf{F} = z \, \mathbf{i} + x \, \mathbf{j} + y \, \mathbf{k}$. This vector field is sort of illustrated in Figure 1c: all I've done is plot the values of the vector field on the xy-, xz-, and yz-planes. (Otherwise, this would be chaos. The effect of this vector field is a rotation about the line $x = y = z$, together with motion along that line away from the origin.

We can see how the rotation part of this description is accurate by computing the curl

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

The vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ indicates the axis of rotation, and it is parallel to the line $x = y = z$.

This makes the curl integral easier to take, because the curl is constant. We have

$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,\mathrm{d}S = \int_{u=0}^{1}\int_{v=0}^{2\pi}(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(3u\cos v\,\mathbf{i}+3u\sin v\,\mathbf{j}+u\,\mathbf{k})\,\mathrm{d}v\,\mathrm{d}u$$

which simplifies to the integral

$$\int_{u=0}^{1}\int_{v=0}^{2\pi}(3u\cos v+3u\sin v+u)\,\mathrm{d}v\,\mathrm{d}u.$$

Both $\sin v$ and $\cos v$ are being integrated over an entire period, so the result cancels. Only the integral of u matters, and we get

$$\int_{u=0}^{1}\int_{v=0}^{2\pi}u\,\mathrm{d}v\,\mathrm{d}u = \int_{u=0}^{1}2\pi u\,\mathrm{d}u = \pi u^2\Big|_{u=0}^{1} = \pi.$$

What about the circulation integral? Here, we evaluate $\mathbf{F} = z\,\mathbf{i}+x\,\mathbf{j}+y\,\mathbf{k}$ at $\mathbf{r}(1,t) = (\cos t, \sin t, 0)$ and get

$$\mathbf{F}(\mathbf{r}(1,t)) = \cos t\,\mathbf{j}+\sin t\,\mathbf{k}.$$

The derivative of $\mathbf{r}(1,t)$ is $-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}$, and the dot product of the two vectors is just $\cos^2 t$. So we get

$$\iint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{t=0}^{2\pi}\cos^2 t\,\mathrm{d}t.$$

We've looked at this integral several times, but here's a way to evaluate this integral we haven't tried yet.[2] If we shift the function $\cos^2 t$ by $\pi/2$, we get $\sin^2 t$. But shifting the function by $\pi/2$ shouldn't change its integral over the entire period of $[0, 2\pi]$: we just move a segment of length $\pi/2$ from one end of the integral to the other. So it must be true that

$$\int_{t=0}^{2\pi}\cos^2 t\,\mathrm{d}t = \int_{t=0}^{2\pi}\sin^2 t\,\mathrm{d}t.$$

But if both of these are equal, then their sum must be twice the circulation integral we want! And their sum is just the integral of $\cos^2 t+\sin^2 t$: the integral of 1 over $[0, 2\pi]$, which is 2π. Therefore π (half of 2π) is the value of the circulation integral. We've confirmed that on this one example, Stokes' theorem holds!

2 From boundary to surface

That wasn't really *applying* Stokes' theorem, just verifying it. Here's an interesting application of Stokes' theorem to do a difficult circulation integral.

To illustrate my point, I will pick a vector field which is very complicated, but whose curl is simple: the vector field

$$\mathbf{F} = x^{5/7}\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}+e^{-z^2/2}\,\mathbf{k}$$

[2] This takes a bit to explain, but is easy to do once you understand the explanation.

whose curl is $\nabla \times \mathbf{F} = 2x\,\mathbf{k}$.

For our curve, let's take the curve C parameterized by $\mathbf{r}(t) = (\cos t, \sin t, \cos 2t)$, where $t \in [0, 2\pi]$.

We only have a curve—not a surface. What do we do? The trick is to come up with a nice surface that will have this curve as its boundary.

One way to do this is to remember the double-angle formula $\cos 2t = \cos^2 t - \sin^2 t$. Writing $\mathbf{r}(t)$ as $(\cos t, \sin t, \cos^2 t - \sin^2 t)$, we can see that the curve lies on the surface $z = x^2 - y^2$. So now we have a description for the surface S whose boundary is C: it is the surface which has equation $z = x^2 - y^2$ and lies above the unit disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

(Where did the unit disk come from? From looking at the x- and y-coordinates of $\mathbf{r}(t)$: those go around the unit circle. So S must stay within that circle.)

By using the implicit form of the surface integral, we get that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S = \iint_D \frac{(\nabla \times \mathbf{F}) \cdot \nabla f}{\nabla f \cdot \mathbf{k}}\,\mathrm{d}x\,\mathrm{d}y$$

where $f(x, y, z) = z - x^2 + y^2$. We have $\nabla f = -2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}$, and $\nabla f \cdot \mathbf{k} = 1$. Furthermore, $(\nabla \times \mathbf{F}) \cdot \nabla f$ simplifies to $2x$.

So we must integrate

$$\iint_D 2x\,\mathrm{d}x\,\mathrm{d}y.$$

The region D is symmetric across the x-axis: it is the unit disk. The function $2x$ is an odd function: $2(-x) = -2x$. Therefore the integral simplifies to 0.

You might ask: what if we had used a different double-angle formula, such as $\cos 2t = 2\cos^2 t - 1$, or $\cos 2t = 1 - 2\sin^2 t$? The answer is that we would have gotten a different surface S with the same boundary C. Stokes' theorem tells us, in particular, that whenever two surfaces have the same boundary like this, the curl integrals of any vector field over those surfaces are equal.

3 A proof of Stokes' theorem

Let's end the day by seeing a proof of Stokes' theorem—or at least, a sketch of a proof.

3.1 Planes parallel to the xy-plane

First, we can start with Green's theorem to conclude that Stokes' theorem holds for regions in the xy-plane. This is essentially the same as the motivational example we started with in the previous lecture.

There is a caveat to this: Green's theorem was only stated for vector fields $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, where M and N are functions of x and y. In our three-dimensional setting, we look at vector fields $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, where M, N, and P are functions of x, y, and z. However, Green's theorem still applies in the more general setting, due to a combination of two factors:

- When we are working with the xy-plane, z is set to a constant: namely, 0. So it will not affect either side of the equation of Stokes' theorem (or of Green's theorem).
- The $\mathbf{k}$-component $P\,\mathbf{k}$ of $\mathbf{F}$ will not interfere. It does not affect the circulation integral, which we can write as

$$\int_C \mathbf{F} \cdot \mathbf{T}\,\mathrm{d}s,$$

because the tangent vector $\mathbf{T}$ is a vector in the xy-plane, so the dot product will multiply P by 0. Also, the $\mathbf{k}$-component of $\mathbf{F}$ will not affect the curl integral, because it is only used to determine the $\mathbf{i}$- and $\mathbf{j}$-components of the curl, $\boldsymbol{\nabla} \times \mathbf{F}$, and we then take the dot product $(\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{k}$.

In fact, the same exact argument will show that Green's theorem holds for regions in any plane *parallel* to the xy-plane (where z is still set to a constant, but not necessarily 0).

3.2 Stokes' theorem for flat regions

There is only a small step from this to proving Stokes' theorem for *all* flat regions. The idea that makes this work is that any plane in $\mathbb{R}^3$ can be reduced to the case above by a linear uvw-substitution: we can choose the parameters of the transformation so that the equation of the plane in uvw-coordinates is just something like "$w = c$". The description of our surface, in uvw-coordinates, is just going to be saying that $w = c$, and adding that the variables (u, v) lie in some region R in the uv-plane.

We can use the uvw-substitution to define a parameterization of our surface S: this will be a "wacky-domain" substitution defined by setting $w = c$ and letting (u, v) vary over the domain R. I will leave out the details of saying that the three-dimensional curl integral over S turns into a two-dimensional curl integral over R. The nontrivial idea here is that the curl $\boldsymbol{\nabla} \times \mathbf{F}$ will transform reasonably after a change of coordinates—it is a basis-independent measure of the rotation. We don't have the machinery in this class to do that neatly, and doing it messily would just involve lots of linear algebra with determinants and the multivariate chain rule.

From there, the idea is to transform the two-dimensional curl integral over R into a circulation integral, by applying Green's theorem. Then, we can realize that this is actually equal to the three-dimensional curl integral over the boundary of S.

I am describing all this somewhat technically, but the intuitive idea is simply that any plane, viewed from the right angle, looks exactly like the xy-plane. If we believe that our definitions of the curl and circulation integrals are natural ones, they shouldn't depend on our point of view. That means that any fact that holds for the xy-plane should hold for *any* plane we choose.

3.3 Stokes' theorem for polyhedral regions

To go from flat regions to arbitrary regions, we can take a polyhedral approximation of an arbitrary surface. As we've done many times before, we first divide our surface into many small cells; then, we approximate each cell by a small flat surface.

Mathematicians don't often think about this process beyond saying "take the limit", but video game developers do: in almost any video game with three-dimensional graphics, all the surfaces you see *are* always being approximated by polygons! Video game developers have a clever idea to make sure this process works: instead of subdividing into square cells, subdivide into triangular cells. Any three points in space determine a plane. So if you divide a curved surface into many triangular regions, you can then approximate each curved triangular region by a flat triangle.

On each flat triangle, Stokes' theorem holds: the curl integral over the triangle is equal to the circulation integral over its boundary. Then, we put both kinds of integrals together:

- For the curl integrals, we can just straightforwardly say that the integrals are additive: their sum is the curl integral over the entire (polyhedral) surface we get by putting all the triangles together.
- For the circulation integrals, what will happen is that—provided the orientations of the triangles are consistent—the circulation will cancel along the boundary between two neighboring triangles. (We saw this happen in the proof of Green's theorem.)

  The only circulation left over will be along segments that are the boundary of only one triangle—and those are precisely the segments that make up the boundary of our polyhedral surface.

We conclude that Stokes' theorem is *definitely* true for polyhedral surfaces made up of flat triangles. Then, we must take the limit to conclude that Stokes' theorem holds in general; this involves arguing that neither the circulation integral nor the curl integral change too much when you pass from a curved surface to its polyhedral approximation.