Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 26: The divergence theorem

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1 From Green's theorem to the divergence theorem

In the recent lectures, we saw Stokes' theorem: a generalization of Green's theorem to three dimensions. Today, we are going to see the divergence theorem, which is a generalization of Green's theorem to three dimensions.

The idea is that Green's theorem has two interpretations: one for circulation and one for flux. Stokes' theorem is a generalization of the circulation form of Green's theorem. In two dimensions, circulation around a simple closed curve is equal to a curl integral over its interior: that curl integral is a scalar double integral, because in two dimensions, curl is a scalar. In three dimensions, circulation around a simple closed curve is equal to a curl integral over a surface with that curve as its boundary: that curl integral is a vector surface integral, because in three dimensions, curl is a vector.

Meanwhile, the divergence theorem is a generalization of the flux interpretation of Green's theorem. In two dimensions, flux across a simple closed curve is equal to a divergence integral over its interior. In three dimensions, it wouldn't even make sense to talk about flux across a curve! Flux measures how much a vector field crosses a boundary—and in three dimensions, a boundary is a surface, enclosing a three-dimensional solid region.

Let's be more concrete. In two dimensions, a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ has divergence $\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$, measuring how much $\mathbf{F}$ is expanding at a point. In three dimensions, the divergence of a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is given by a very similar formula: $\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$. A different notation is also popular. If $\boldsymbol{\nabla}$ is the operator $\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$, then the dot product $\boldsymbol{\nabla}\cdot\mathbf{F}$ is going apply $\frac{\partial}{\partial x}$ to the $\mathbf{i}$-component of $\mathbf{F}$, apply $\frac{\partial}{\partial y}$ to the $\mathbf{j}$-component of $\mathbf{F}$, apply $\frac{\partial}{\partial z}$ to the $\mathbf{k}$-component of $\mathbf{F}$, and add them all up. So $\boldsymbol{\nabla}\cdot\mathbf{F}$ is another way to write $\operatorname{div}\mathbf{F}$.

In two dimensions, Green's theorem says that

$$\int_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \iint_R \operatorname{div}\mathbf{F}\,\mathrm{d}A = \iint_R \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}A,$$

where R is a region in $\mathbb{R}^2$ whose boundary is C.

In three dimensions, we take a solid region R whose boundary is the surface S. We replace the flux integral of Green's theorem by a flux integral over S, and the divergence integral of Green's theorem by a divergence integral over R. The divergence theorem says that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_R \operatorname{div}\mathbf{F}\,\mathrm{d}V = \iiint_R \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V.$$

[1] This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

The flux integral across S turns into an ordinary triple integral!

A note on orientation: as we know, the flux integral across S is an oriented integral, whose sign depends on the orientation we give S. In many cases, this is a pain. In the case of closed surfaces S, which are the boundary of a solid region R, there is a conventional choice of orientation: "outward". Just as with Green's theorem, it is the outward flux across S to which the divergence theorem applies.

(Just as with Green's theorem, the divergence theorem can be stated in terms of differential forms; in that case, the orientation match is more subtle, and the surface integral across S with any orientation will turn into an oriented volume integral over R. We will learn more about this in the next lecture.)

2 An example of the divergence theorem

2.1 Flux out of a cylinder

Let's look at an example. Let R be the cylinder bounded by $-1 \le z \le 1$ and $x^2 + y^2 \le 1$, and let $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k}$. What is the outward flux of $\mathbf{F}$ across the boundary of R?

The direct approach. The boundary consists of three surfaces: S_1 is the top circle, S_2 is the lateral surface of the cylinder, and S_3 is the bottom circle.

The surface S_1 lies in the plane $z = 1$, so it has a constant normal vector of $\mathbf{k}$, and $\mathbf{F} \cdot \mathbf{k} = z^2 = 1$. Therefore

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_{S_1} 1\,\mathrm{d}S$$

which is the area of S_1: it is π. Similarly, the surface S_3 lies in the plane $z = -1$, and the normal vector pointing outward from R is $-\mathbf{k}$; we get $\mathbf{F} \cdot -\mathbf{k} = -z^2 = -1$. Therefore

$$\iint_{S_3} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_{S_3} -1\,\mathrm{d}S$$

which is the negative of the area of S_3: it is $-\pi$, canceling out the first integral. Finally, if we parameterize S_2 by $\mathbf{r}(u,v) = (\cos u, \sin u, v)$ with $(u,v) \in [0, 2\pi] \times [-1, 1]$, then

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (-\sin u\,\mathbf{i} + \cos u\,\mathbf{j}) \times \mathbf{k} = \cos u\,\mathbf{i} + \sin u\,\mathbf{j}$$

and $\mathbf{F}(\mathbf{r}(u,v)) = (\cos^2 u + \sin^2 u)\,\mathbf{i} + \sin u\,\mathbf{j} + v^2\,\mathbf{k} = \mathbf{i} + \sin u\,\mathbf{j} + v^2\,\mathbf{k}$. We get

$$\begin{aligned}
\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S &= \int_{u=0}^{2\pi} \int_{v=-1}^{1} \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \mathrm{d}v\,\mathrm{d}u \\
&= \int_{u=0}^{2\pi} \int_{v=-1}^{1} (\mathbf{i} + \sin u\,\mathbf{j} + v^2\,\mathbf{k}) \cdot (\cos u\,\mathbf{i} + \sin u\,\mathbf{j})\,\mathrm{d}v\,\mathrm{d}u \\
&= \int_{u=0}^{2\pi} \int_{v=-1}^{1} (\cos u + \sin^2 u)\,\mathrm{d}v\,\mathrm{d}u \\
&= 2 \int_{u=0}^{2\pi} (\cos u + \sin^2 u)\,\mathrm{d}u = 2 \left(\sin u + u - \frac{1}{2} \sin 2u \right) \Bigg|_{u=0}^{2\pi} = 2\pi.
\end{aligned}$$

Altogether, the outward flux is $\pi + 2\pi + -\pi = 2\pi$.

The first approach is just there for review, and so we can better appreciate the simplicity of the second approach:

Using the divergence theorem. The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2) = 2x + 1 + 2z.$$

To integrate over R, we write this in cylindrical coordinates:

$$\begin{aligned}
\iiint_R (2x + 1 + 2z)\,\mathrm{d}V &= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=-1}^{1} (2r\cos\theta + 1 + 2z) r \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (4r^2\cos\theta + 2r)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi} \left(\frac{4}{3}\cos\theta + 1\right) \mathrm{d}\theta \\
&= \frac{4}{3}\sin\theta + \theta \Big|_{\theta=0}^{2\pi} = 2\pi.
\end{aligned}$$

2.2 Flux out of a sphere

We can look at a simpler example. What is the outward flux of

$$\mathbf{F} = 4xyz\,\mathbf{i} - y^2 z\,\mathbf{j} - yz^2\,\mathbf{k}$$

across the unit sphere?

With the divergence theorem, this would be an integral over the interior of the unit sphere, but it is an integral of the divergence of $\mathbf{F}$: of

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(4xyz) + \frac{\partial}{\partial y}(-y^2 z) + \frac{\partial}{\partial z}(-yz^2) = 4yz - 2yz - 2yz = 0.$$

So the flux of $\mathbf{F}$ across the unit sphere—or, indeed, the boundary of any other region—is 0.

(In this example, it turns out that $\mathbf{F}$ is the curl $\nabla \times \mathbf{G}$, when $\mathbf{G} = -xyz^2\,\mathbf{i} + xy^2 z\,\mathbf{k}$. This gives another reason to believe that the flux of $\mathbf{F}$ across the unit sphere is 0: by Stokes' theorem, it is the circulation of $\mathbf{G}$ around the boundary of that surface, but the sphere *has* no boundary.)

3 Regions with holes

The divergence theorem, in its simplest and most direct version, only applies to solid regions with a single boundary. What if the region R has holes in it?

Well, we have to distinguish between different types of holes. The hole in a torus (or donut) is a one-dimensional hole, and it does not get in the way of us applying the divergence theorem. A solid torus still has a single boundary, we can still consider the outward flux across that boundary, and it is still equal to the integral of the divergence over the solid torus.

The holes that interfere with the divergence theorem are a different kind: they are internal cavities completely enclosed by the region, like the air pockets in Swiss cheese. For a concrete example, take a solid sphere of radius 2, with a solid sphere of radius 1 removed from it: the region $R = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 4\}$.

What does the divergence theorem say about the region R? Directly, nothing. Indirectly, we can still argue that for any vector field $\mathbf{F}$, we have

$$\iiint_R \boldsymbol{\nabla} \cdot \mathbf{F}\,\mathrm{d}V = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S,$$

where:

- S_1 is the outer boundary of R: it is the sphere with equation $x^2 + y^2 + z^2 = 4$, oriented with a normal vector pointing outward.
- S_2 is the inner boundary of R: it is the sphere with equation $x^2 + y^2 + z^2 = 1$, oriented with a normal vector pointing *inward*.

We can justify this calculation in two ways.

First, we can go back to our description of R as "a solid sphere of radius 2, with a solid sphere of radius 1 removed from it". If we *didn't* remove the core of this solid sphere, then the divergence integral would simply be the outward flux across S_1. To remove that core, we subtract the divergence integral over the solid sphere of radius 1. This is the same as subtracting the outward flux across S_2, by the divergence theorem—or adding the *inward* flux across S_2.

Second, we can take R and chop it in half, for example along the plane $z = 0$. Since we're in Georgia, picture cutting a peach in half, and removing the pit: the two halves are precisely the two portions of R.

Both halves of the peach are regions with no more holes in them. Topologically, each has a single boundary, it's just that the boundary has several parts with different shapes.

- For the top half of the peach, the boundary consists of the top half of the slightly-fuzzy skin of the peach (oriented outward), the plane where you cut the peach (oriented downward), and the cavity where you removed the pit (oriented inward, that is with normal vector pointing toward the origin).
- For the bottom half of the peach, the boundary consists of the top half of the slightly-fuzzy skin of the peach (oriented outward), the plane where you cut the peach (oriented upward), and the cavity where you removed the pit (oriented inward).

When we take the flux integrals across these two regions and add them together:

- Both regions contribute an outward flux across the skin of the peach: this is the flux integral across S_1.
- Both regions contribute a flux across a surface in the plane $z = 0$, where you cut the peach, but oriented in opposite directions; these contributions cancel.
- Both regions contribute an inward flux into the cavity where you removed the pit: this is the flux integral across S_2.

The second justification is a bit more complicated, but it remains valid for a vector field $\mathbf{F}$ which is undefined at the origin, for example.

4 Surfaces with a boundary

The divergence theorem only applies to surfaces without a boundary, because the surface we use must itself be the boundary of a solid region. However, we can work around that in some cases.

Let S_1 be a hyperbolic surface: the portion of the graph of $z = xy$ with $-1 \le x \le 1$ and $-1 \le y \le 1$, given an upward orientation. For many vector fields $\mathbf{F}$, the flux of $\mathbf{F}$ across S_1 might be a tricky surface integral to take.

However, there is a different surface with the same boundary: the surface $z = |x+y| - 1$ with $-1 \le x \le 1$ and $-1 \le y \le 1$. We can break it up as the union $S_2 \cup S_3$, where S_2 has equation $z = -x - y - 1$ and covers the part with $x + y \le 0$, and S_3 has equation $z = x + y - 1$ and covers the part with $x + y \ge 0$. Together, S_1, S_2, and S_3 form the boundary of a solid region R. If we orient S_2 and S_3 with normal vector pointing *downward*, then we have

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S + \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_R \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V.$$

The flux integrals across S_2 and S_3 may turn out to be simpler to evaluate, because S_2 and S_3 each lie in a flat plane. For S_2, writing the plane as $x + y + z = -1$, we get a normal vector of $\mathbf{i}+\mathbf{j}+\mathbf{k}$; we normalize it and orient it downward into $-\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$. For S_3, writing the plane as $x + y - z = 1$, we get a normal vector of $\mathbf{i}+\mathbf{j}-\mathbf{k}$; we normalize it and orient it downward into $\frac{\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{3}}$.

To take a triple integral over R, we also divide it into two cases, with $x + y \le 0$ and $x + y \ge 0$. We integrate from S_2 to S_1 in the first case and from S_3 to S_1 in the second:

$$\begin{aligned}\iiint_R \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V &= \int_{x=-1}^{1}\int_{y=-1}^{-x}\int_{z=-x-y-1}^{xy} \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_{x=-1}^{1}\int_{y=-x}^{1}\int_{z=x+y-1}^{xy} \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

If we do not want to take a flux integral across S_1, we can instead take four integrals: a flux integral across S_2, a flux integral across S_3, and the two triple integrals that make up the divergence integral over R.

Is it worth it? It's hard to say. Here's a vector field $\mathbf{F}$ for which we would benefit from this approach a lot: $\mathbf{F} = e^z\,\mathbf{i} - e^z\,\mathbf{j} + xyz\,\mathbf{k}$. The surface integrals over S_2 and S_3 are going to be very simple here, because $\mathbf{F}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = xyz$ and $\mathbf{F}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k}) = -xyz$. The divergence $\boldsymbol{\nabla}\cdot\mathbf{F}$ also simplifies considerably: it is just xy.