Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 27: More about the divergence theorem

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1 The divergence theorem with differential forms

1.1 The exterior derivative of a 2-form

In the language of differential forms, we think of the surface integral of a vector field $M\,\mathbf{i}+N\,\mathbf{j}+P\,\mathbf{k}$ as an integral of the 2-form

$$\omega = M\,\mathrm{d}y \wedge \mathrm{d}z + N\,\mathrm{d}z \wedge \mathrm{d}x + P\,\mathrm{d}x \wedge \mathrm{d}y.$$

The divergence theorem can be stated in terms of the exterior derivative $\mathrm{d}\omega$. This is defined in the same way as we defined the exterior derivative of functions and of 1-forms:

$$\mathrm{d}\omega = \mathrm{d}M \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}N \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}P \wedge \mathrm{d}x \wedge \mathrm{d}y.$$

But how does this simplify? We have $\mathrm{d}M = \frac{\partial M}{\partial x}\,\mathrm{d}x + \frac{\partial M}{\partial y}\,\mathrm{d}y + \frac{\partial M}{\partial z}\,\mathrm{d}z$, but by the rule that $\mathrm{d}u \wedge \mathrm{d}u = 0$, $\mathrm{d}M \wedge \mathrm{d}y \wedge \mathrm{d}z$ only keeps the first of these three terms: it is equal to $\frac{\partial M}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. Similarly, $\mathrm{d}N \wedge \mathrm{d}z \wedge \mathrm{d}x$ simplifies to $\frac{\partial N}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x$, and $\mathrm{d}P \wedge \mathrm{d}x \wedge \mathrm{d}y$ simplifies to $\frac{\partial P}{\partial z}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y$.

These are all 3-forms, and with 3-forms, it looks like there are many orders for the differential elements in the wedge product. (Previously, with 2-forms, we only had to deal with $\mathrm{d}x \wedge \mathrm{d}y$ and $\mathrm{d}y \wedge \mathrm{d}x = -(\mathrm{d}x \wedge \mathrm{d}y)$.) However, the rule that $\mathrm{d}u \wedge \mathrm{d}v = -(\mathrm{d}v \wedge \mathrm{d}u)$ lets us turn all triple wedge products of $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ into either $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ or its negation:

- When we encounter the term $\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x$, we can rewrite it as $-(\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z)$ by swapping the last two differentials, and then as $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ by swapping the first two.
- Similarly, $\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y$ turns into $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ by first swapping the first two differentials, then swapping the last two.
- On the other hand, $\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y$, $\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z$, and $\mathrm{d}z \wedge \mathrm{d}y \wedge \mathrm{d}x$ all simplify to $-(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$.

In particular, in our formula for $\mathrm{d}\omega$, all three terms have a triple wedge product that simplifies to $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, so we get

$$\mathrm{d}\omega = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

The exterior derivative of a 2-form computes the divergence of the corresponding vector field!

This lets us write the divergence theorem (for a solid region D with boundary S)

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_D \boldsymbol{\nabla} \cdot \mathbf{F}\,\mathrm{d}V$$

[1]This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

in the by-now-familiar form

$$\iint_S \omega = \iiint_D \mathrm{d}\omega$$

where $\omega = M\,\mathrm{d}y \wedge \mathrm{d}z + N\,\mathrm{d}z \wedge \mathrm{d}x + P\,\mathrm{d}x \wedge \mathrm{d}y$.

There is one difference. Thus far, we've stated the divergence theorem as saying that the *outward* flux across S is equal to the divergence integral over D. The differential form of the divergence theorem does not specify that the flux must be an outward flux; instead, it turns both integrals into oriented integrals.

If we then adopt the convention that an oriented integral of $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is positive, that corresponds to choosing an orientation for D, so we must choose a compatible orientation for S, and the compatible orientation for S is the outward orientation. This is a bit confusing to think about, because it's not natural for us to think of giving $\mathbb{R}^3$ an orientation—four-dimensional mathematicians who could look at $\mathbb{R}^3$ from the outside would find that easier!

1.2 Divergence and curl

Remember the general result we proved for differential forms that $\mathrm{d}(\mathrm{d}\omega) = 0$? If we apply it when $\omega = f$, a function (or 0-form), then $\mathrm{d}f$ corresponds to taking the gradient of f, and $\mathrm{d}(\mathrm{d}f)$ corresponds to taking the curl of that gradient. In this way, we obtain the rule that

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = \mathbf{0}.$$

The curl of a gradient is the zero vector! (This is the "component test" for seeing if a vector field could be a gradient field.)

But we can apply the same rule $\mathrm{d}(\mathrm{d}\omega)$ to the 1-form $\omega = M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$. In this case, $\mathrm{d}\omega$ corresponds to taking the curl of $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, and $\mathrm{d}(\mathrm{d}\omega)$ corresponds to taking the divergence of that curl. In this way, we obtain the rule that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0.$$

The divergence of a curl is zero!

We can think of this, too, as a test for whether a vector field is the curl of another vector field. Such a vector field is special, because it means that:

- The flux of such a vector field across the boundary of any solid region is 0, because the divergence theorem turns that flux integral into an integral of 0 over the solid region.
- Given two surfaces with the same boundary, the fluxes across the two surfaces must be the same, since $\mathbf{F}$ being the curl of another vector field means that Stokes' theorem applies to it.

There are many terms for vector fields with divergence 0. They are naturally called **divergence-free** vector fields. Because divergence measures compression and/or expansion at a point, they are also called **incompressible** vector fields—and the velocity fields of incompressible substances naturally have this property. Wikipedia also tells me that they are called **solenoidal** vector fields, for reasons I do not know. Of course, if $\mathbf{F}$ is the curl of another vector field, we can call $\mathbf{F}$ a **curl** vector field and be understood. Finally, in the language of differential forms, we can talk about

exact 2-forms (which are the exterior derivative, or curl, of some 1-form) and about **closed** 2-forms (whose exterior derivative, or divergence, is 0).

As before, checking whether $\mathbf{F}$ is a curl vector field by checking whether $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ is a one-way test, in general. If we get a nonzero value, then $\mathbf{F}$ is definitely *not* a curl vector field. If we get a zero value, then we quibble and hem and haw and there are weird counterexamples with undefined points. However, provided that the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$, then the test is an if-and-only-if condition.

2 A proof of the divergence theorem

2.1 Division into pieces

An important observation (which we made in special cases in the previous lecture) is the following. Suppose that a solid region $D \subseteq \mathbb{R}^3$ is the union $D_1 \cup D_2$ of two smaller regions, and that D_1 and D_2 either do not intersect at all, or else their intersection has zero volume. (For example, if we chop a solid sphere in half, the two halves intersect along a circle in the plane where we chopped; that's an intersection with zero volume.)

In this case, at least when D, D_1, and D_2 are all reasonable regions, we have

$$\iiint_D \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V = \iiint_{D_1} \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V + \iiint_{D_2} \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V \tag{1}$$

for any vector field $\mathbf{F}$. In fact, a version of (1) should hold no matter what we integrate, even if it's not the divergence of anything.

Let S, S_1, and S_2 be the outward-oriented surfaces bounding D, D_1, and D_2, respectively. Then the divergence theorem, applied to every integral in (1), tells us that

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S + \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S. \tag{2}$$

In fact, if we're going to be proving the divergence theorem, it will be convenient for us to establish that (2) holds *without* using that theorem—because we would like to use (2) in the proof!

If the regions D_1 and D_2 are far away from each other and don't touch at all (in which case, D is made up of two separated pieces), then there's nothing to check, really. The integrals on the right-hand side of (2) are just two completely separate parts of the integral on the left-hand side.

Something more subtle happens when D_1 and D_2 share a boundary: when they touch along an entire surface $S_1 \cap S_2$. In that case, both integrals on the right-hand side of (2) include flux across $S_1 \cap S_2$, but the left-hand side does not include it at all. (If D_1 and D_2 touch along $S_1 \cap S_2$, then some part of D is present on both sides of $S_1 \cap S_2$, which means that $S_1 \cap S_2$ is not a *boundary* of D.)

What saves us in this case is orientation. All three flux integrals in (2) are outward-oriented, but for the intersection $S_1 \cap S_2$, "outward" is context-sensitive. On one side of $S_1 \cap S_2$ lies region D_1; on the other side, region D_2. In the outward flux integral across S_1, we measure the D_1-to-D_2 flux of $\mathbf{F}$; in the outward flux integral across S_2, we measure the D_2-to-D_1 flux. These are equal but opposite quantities; when we add together the integrals, they cancel. Thus, the flux across $S_1 \cap S_2$ makes no net contribution to either side of (2), and the equation still holds.

2.2 A proof using stripes

There are two *styles*, so to speak, to proofs of these differential theorems from scratch. When we proved Green's theorem, we took one of the main approaches: we divided our region into cells, and looked at each cell individually. (When we proved Stokes' theorem, we took the lazy way out and appealed to Green's theorem.)

Here, we will look at a different style of proof; instead of dividing into cells, we will divide into *columns*. We could have used this technique to prove Green's theorem, and we could use our previous technique to prove the divergence theorem, but variety is the spice of life.

First, we simplify the problem. Instead of proving the theorem for all vector fields in general, it's enough to prove it for $\mathbf{F} = P\,\mathbf{k}$: vector fields with $M = N = 0$. The same technique with the variables permuted will apply to vector fields of the form $M\,\mathbf{i}$ or $N\,\mathbf{j}$. Finally, once we have those cases handled, we can just add all three types together, because both sides of the divergence theorem are linear.

Second, we simplify the problem. Instead of looking at arbitrary regions, we will look at regions D bounded from below by a surface S^- that lies on the graph of $f^-(x, y, z) = 0$, from above by a surface S^+ that lies on the graph of $f^+(x, y, z) = 0$, and constrained on the sides by the condition $(x, y) \in R$ for some region $R \subseteq \mathbb{R}^2$. Not all regions have this form—but sufficiently nice regions can be chopped up into several pieces that have this form. You should imagine the cylinder with $x^2 + y^2 \le 1$ and $-1 \le z \le 1$ for a simple example; here, R is the unit disk in the xy-plane, S^- is the portion of the plane $z = -1$ below this disk, and S^+ is the portion of the plane $z = 1$ above this disk.

The boundary of this region has three parts to it: the surface S^-, oriented downward; the surface S^+, oriented upward; and the lateral boundary. We can ignore the lateral boundary, however, because our vector field $\mathbf{F} = P\,\mathbf{k}$ is tangent to it, so it has zero flux across it.

For the surface S^+, oriented upward, we can write the flux integral as

$$\iint_{S^+} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_R \frac{\mathbf{F} \cdot \boldsymbol{\nabla} f^+}{\boldsymbol{\nabla} f^+ \cdot \mathbf{k}}\,\mathrm{d}A = \iint_R \frac{P\,\mathbf{k} \cdot \boldsymbol{\nabla} f^+}{\boldsymbol{\nabla} f^+ \cdot \mathbf{k}}\,\mathrm{d}A = \iint_R P\,\mathrm{d}A.$$

For the surface S^-, oriented downward, we can write the flux integral as

$$\iint_{S^-} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_R \frac{\mathbf{F} \cdot \boldsymbol{\nabla} f^+}{-\boldsymbol{\nabla} f^+ \cdot \mathbf{k}}\,\mathrm{d}A = \iint_R \frac{P\,\mathbf{k} \cdot \boldsymbol{\nabla} f^+}{\boldsymbol{\nabla} f^+ \cdot \mathbf{k}}\,\mathrm{d}A = \iint_R -P\,\mathrm{d}A.$$

We add these two flux integrals and get 0, proving that all flux integrals are 0... just kidding!

The problem is with our notation. In both of these flux integrals, P is a function of x, y, and z, which is meant to be evaluated on the surface S^+ or S^-, respectively. If we rewrite $f_\pm(x, y, z) = 0$ as $z = h_\pm(x, y)$, solving for z, then the proper way to write what we've just done is

$$\iint_{S^+} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_R P(x, y, h^+(x, y))\,\mathrm{d}A \quad \text{and} \quad \iint_{S^-} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_R -P(x, y, h^-(x, y))\,\mathrm{d}A.$$

Their sum, the net flux out of D, is the integral

$$\iint_R \big(P(x,y,h^+(x,y)) - P(x,y,h^-(x,y))\big)\,\mathrm{d}A.$$

Now we apply the fundamental theorem of calculus to say that

$$P(x,y,h^+(x,y)) - P(x,y,h^-(x,y)) = \int_{z=h^-(x,y)}^{h^+(x,y)} \frac{\partial P}{\partial z}(x,y,z)\,\mathrm{d}z.$$

This lets us rewrite our net flux out of D as

$$\iint_{(x,y)\in R} \int_{z=h^-(x,y)}^{h^+(x,y)} \frac{\partial P}{\partial z}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

which is *precisely* the divergence integral of $\mathbf{F}$ over D! The divergence of $\mathbf{F} = P\,\mathbf{k}$ is just $\frac{\partial P}{\partial z}$, and to integrate over D with z as the variable of the innermost integral, we let (x,y) range over R and then let z range from $h^-(x,y)$ to $h^+(x,y)$.

So we've shown that the net flux of $\mathbf{F}$ out of D is equal to the integral of the divergence of $\mathbf{F}$ over D, which proves the divergence theorem in our special case—and, as we argued earlier, this is enough to conclude that it holds in general.

3 The divergence theorem and volume

Just as we can use Green's theorem to compute the area of a region in $\mathbb{R}^2$, we can use the divergence theorem to compute the volume of a region in $\mathbb{R}^3$. For this, we cook up a vector field $\mathbf{F}$ such that $\boldsymbol{\nabla}\cdot\mathbf{F} = 1$. There are many possibilities here: we can add any divergence-free vector field to $\mathbf{F}$, and it will not change the value of $\boldsymbol{\nabla}\cdot\mathbf{F}$. Two natural choices are:

- $\mathbf{F} = x\,\mathbf{i}$; then $\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}x = 1$.
- $\mathbf{F} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{3}$; then $\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}\frac{x}{3} + \frac{\partial}{\partial y}\frac{y}{3} + \frac{\partial}{\partial z}\frac{z}{3} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$.

The first has the advantage of simplicity; the second has the advantage of symmetry.

In both cases, for any solid region D with boundary S, we can compute

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_D \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iiint_D \mathrm{d}V$$

and get the volume of D. When $\mathbf{F} = x\,\mathbf{i}$, a convenient way to write this is

$$\iiint_D \mathrm{d}V = \iint_S x\,\mathrm{d}y\wedge\mathrm{d}z.$$

We can also be more general. If $\boldsymbol{\nabla}\cdot\mathbf{F}$ is any constant c, then the flux of $\mathbf{F}$ across S will give c times the volume of D.

For example, let's use this to compute the volume of a cone. Our cone will have height H and radius R (I am using capital letters to indicate that these are constants, to avoid confusion with the variable r). Its base is a circle of radius R in the xy-plane centered at $(0,0)$ and its vertex is at $(0,0,H)$.

It is the *flexibility of choice* of vector field $\mathbf{F}$ that makes the divergence theorem more powerful as a tool here. If we were to use $x\,\mathbf{i}$, $y\,\mathbf{j}$, or $z\,\mathbf{k}$ as our vector field in this example, we would be back to the ordinary computation for the volume of a cone by triple integral. Instead, let's be sneaky and choose $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + (z - H)\,\mathbf{k}$, with $\boldsymbol{\nabla} \cdot \mathbf{F} = 3$.

Why did we choose this vector field $\mathbf{F}$? Because, for any point (x, y, z), $\mathbf{F}$ is the displacement vector that points from $(0,0,H)$ to (x, y, z). If we look at a point on the lateral surface of our cone, this displacement vector is tangent to the cone. Therefore the flux of $\mathbf{F}$ across this lateral surface will be 0.

As a result, the flux of $\mathbf{F}$ across the boundary of the cone is entirely equal to the flux across the bottom face: the base of the cone. Here, the normal vector is $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F} \cdot \mathbf{n} = -(z - H) = H - z$, and because the base of the cone lies in the plane $z = 0$, this simplifies to H. We conclude that

$$\iiint_D 3\,\mathrm{d}V = iint_B H\,\mathrm{d}S$$

where D is the cone and B is its base. In other words, 3 times the volume is equal to H times the area of the base. The base has area πR^2, and therefore the volume is $\frac{1}{3}\pi R^2 H$.