Math 3204: Calculus IV[1]	Mikhail Lavrov
Lecture 5: Curves and their parameterizations	
August 28, 2023	*Kennesaw State University*

1 Curves in 2 and 3 dimensions

In the next several lectures, we will look at several kinds of **line integrals**. The name is somewhat misleading. A line integral is not necessarily an integral over a straight line; it is an integral over a **curve** in $\mathbb{R}^2$ or $\mathbb{R}^3$ (or, theoretically, in higher-dimensional Euclidean space).

What is a curve? Informally, it's a one-dimensional set, which has length but no thickness. We will often see curves being used to model the path taken by a moving point. If a physical object is shaped like a wire of negligible thickness, it makes sense to model its shape by a curve. Curves can have a start and end, or they can be **closed**: ending where they start.

In $\mathbb{R}^2$, a single equation is often enough to specify a curve. For example, the unit circle

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is an example of a curve. (Sometimes, however, the graph of a single equations consists of multiple curves, or has infinitely long components, which are "too big" to be curves.)

To work with a curve algebraically, it is useful to have a **parameterization** of a curve. Formally, a parameterization of a curve C in $\mathbb{R}^n$ is a function $\mathbf{r}\colon [a, b] \to \mathbb{R}^n$ whose image is C; here, $[a, b]$ can be any interval of finite length. For example, a standard way to parameterize the unit circle is by the parameterization

$$\mathbf{r}(t) = (\cos t, \sin t), \qquad t \in [0, 2\pi].$$

We'd like this function to be a nice one. At the very least, let's say it's differentiable, and that it's injective except maybe at a few isolated points—we don't want to be *too* strict there, because the parameterization above has $\mathbf{r}(0) = \mathbf{r}(2\pi)$.

Here are two ways to think about the relationship between the function $\mathbf{r}(t)$ and the curve C:

- The variable t represents time; a particle traces the curve C over the time interval $[a, b]$, and at time t, the particle is at point $\mathbf{r}(t)$.
- The variable t is a coordinate that tells us a location on C. (Imagine tiny creatures living on C; when they use GPS, they only need one number to specify their location, and that number is t.) The function $\mathbf{r}$ converts our t-coordinate on C to an xyz-coordinate in space.

The same curve C can have multiple parameterizations; sometimes very different ones!

A curve can have an **orientation**: a choice of direction along the curve. When we choose a parameterization $\mathbf{r}(t)$, we choose an orientation in the process, because "the direction of increasing t" is a choice of direction. Some properties of curves (such as arc length) do not depend on the orientation; other properties will depend on it.

[1]This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

2 Simple parameterizations

Finding a parameterization for a curve is, in general, a hard problem. We will work our way up to complicated curves, starting from simple ideas.

2.1 Functions of x, y, or z

Suppose we graph a function $y = f(x)$ in the plane. A finite portion of this graph (limited by an inequality $a \leq x \leq b$) is a curve. It is a curve that has a very simple parameterization:

$$\mathbf{r}(t) = (t, f(t)), \qquad t \in [a, b].$$

(Why do we even both with a parameterization at all, in this case? Just so that we can work with all curves in the same way—those that are graphs of a function, and those that are not.)

Whenever a curve C (in $\mathbb{R}^2$ or in $\mathbb{R}^3$) has only a single point with each x-coordinate, we can solve for the other coordinates that point in terms of x, and then obtain a parameterization of the curve with $x = t$.

There is nothing special about x; the same can be done with any coordinate. Sometimes we even have a choice! For example, suppose we take the graph of $y = x^2$ in the range $1 \leq x \leq 2$. Then we can parameterize it in two ways:

- By solving for y in terms of x, getting $\mathbf{r}(t) = (t, t^2)$ where $t \in [1, 2]$.
- By solving for x in terms of y, getting $\mathbf{r}(t) = (\sqrt{t}, t)$ where $t \in [1, 4]$.

We may not be given the curve in functional form $y = f(x)$, even if such a form exists. For example, suppose we take the curve

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ and } y \geq 0\}$$

which is the top half of the unit circle. This enables us to solve for y in terms of x: because $y \geq 0$, we can take the square root of both sides in $y^2 = 1 - x^2$ to get $y = \sqrt{1 - x^2}$, and end up with the parameterization

$$\mathbf{r}(t) = (t, \sqrt{1 - t^2}), \qquad t \in [-1, 1].$$

(This is an alternative to the parameterization $\mathbf{r}(t) = (\cos t, \sin t)$ with $t \in [0, \pi]$.)

2.2 Parameterizations inspired by alternate coordinates

What role do coordinate systems like polar, cylindrical, and spherical coordinates play in finding parameterizations?

To begin with: we never want a parameterization *in* one of these alternate coordinate systems: in the end our function $\mathbf{r}(t)$ must always be in rectangular coordinates for us to work with it! (Either $\mathbf{r}(t) = (x(t), y(t))$ or $\mathbf{r}(t) = (x(t), y(t), z(t))$.)

However, we can be *inspired* by such a coordinate system on our way to finding a parameterization. In other words, we can find a parameterization in a different coordinate system as an intermediate step, before converting to rectangular coordinates.

Here is an example. Suppose we want to parameterize a spiral that winds around the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$, climbing up this cylinder at a constant angular rate, starting at the point $(1, 0, 0)$ and ending at the point $(1, 0, 4)$ after making a full turn. How can we do this?

In rectangular coordinates, it would be tricky. (We could try to solve for x and y in terms of z, but this is not as easy as what we will do.) But what are the cylindrical coordinates of a point on this spiral?

- The r-coordinate (not to be confused with $\mathbf{r}$) is always 1, because we stay on the surface of the cylinder $x^2 + y^2 = 1$.
- The θ coordinate varies from 0 to 2π, because the spiral makes one full turn around the cylinder.
- The z-coordinate increases linearly with θ. When $\theta = 0$, $z = 0$. When $\theta = 2\pi$, $z = 4$: that's the height we reach after a full turn. Therefore, in general, $z = \frac{4}{2\pi} \cdot \theta = \frac{2}{\pi} \cdot \theta$.

This means that in cylindrical coordinates, we can write down a parameterization with $\theta = t$. We just set $r = 1$ (a constant) and $z = \frac{2t}{\pi}$.

To get a final answer, we convert back to rectangular coordinates. We have $x = r\cos\theta = \cos t$ and $y = r\sin\theta = \sin t$, getting the parameterization

$$\mathbf{r}(t) = (\cos t, \sin t, \tfrac{2t}{\pi}), \qquad t \in [0, 2\pi].$$

The parameterization of the unit circle in our very first example is another parameterization inspired by another coordinate system: it is a simple parameterization in polar coordinates that we converted to rectangular coordinates.

3 Other useful ideas

3.1 Line segments

There is one specific parameterization that is very useful to know, because it comes up often. How do we parameterize a line segment starting at point $\mathbf{a}$ and ending at point $\mathbf{b}$?

In specific cases, this can often be done by solving for y, z as a function of x, or for x, z as a function of y, or whatever. (If the segment lies flat in one dimension, that can limit our choices.) But there's a useful standard formula for this parameterization:

$$\mathbf{r}(t) = (1 - t)\mathbf{a} + t\mathbf{b}, \qquad t \in [0, 1].$$

For example, to parameterize the line segment from $(1, 0, 3)$ to $(4, 1, -1)$, we take

$$\begin{aligned}\mathbf{r}(t) &= (1 - t)(1, 0, 3) + t(4, 1, -1) \\ &= (1 - t, 0, 3 - 3t) + (4t, t, -t) \\ &= (1 + 3t, t, 3 - 4t)\end{aligned}$$

where $t \in [0, 1]$.

One intuition for this formula is that $(1-t)\mathbf{a} + t\mathbf{b}$ is a weighted average of the points $\mathbf{a}$ and $\mathbf{b}$. When $t = 0$, the weights are 1 and 0, so we get $\mathbf{a}$. When $t = 1$, the weights are 0 and 1, so we get $\mathbf{b}$. When t is between 0 and 1, the weights are somewhere in between, so we get a point between $\mathbf{a}$ and $\mathbf{b}$.

3.2 Piecewise parameterizations

Sometimes a parameterization for the entire curve is difficult to find, but we can break the curve up into several pieces, and parameterize each piece. This often happens, for example, when a curve has completely different behavior in different places.

Suppose we are working in $\mathbb{R}^2$, and we want a counterclockwise parameterization of the boundary of the unit square, going from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$. Each step of this curve can be found by using the formula for line segments, for example. We get four parameterizations for the four sides of the square:

- $\mathbf{r}_1(t) = (t, 0)$, where $t \in [0,1]$.
- $\mathbf{r}_2(t) = (1, t)$, where $t \in [0,1]$.
- $\mathbf{r}_3(t) = (1-t, 1)$, where $t \in [0,1]$.
- $\mathbf{r}_4(t) = (0, 1-t)$, where $t \in [0,1]$.

(Note that we're using $1-t$ rather than t for $\mathbf{r}_3$ and $\mathbf{r}_4$, to preserve the direction around the curve; this will sometimes matter!

In most cases, this is sufficient. For example, when we take a line integral along this curve, we can simply split it up into four parts, and use a different parameterization $\mathbf{r}_i$ on each part.

If we want to have a single function $\mathbf{r}(t)$ that describes all four pieces at once for some reason, this can also be done, with a bit more work. First, we need to rewrite the individual parameterizations to use consecutive intervals: for example, change $\mathbf{r}_2(t)$ from being $(1,t)$ on $[0,1]$ to being $(1, t-1)$ on $[1,2]$. After changing $\mathbf{r}_3$ and $\mathbf{r}_4$ to using intervals $[2,3]$ and $[3,4]$, we can simply combine them all into a piecewise expression:

$$\mathbf{r}(t) = \begin{cases} (t,0) & 0 \le t < 1 \\ (1, t-1) & 1 \le t < 2 \\ (1-(t-2), 1) & 2 \le t < 3 \\ (0, 1-(t-3)) & 3 \le tl34 \end{cases} \qquad t \in [0,4].$$

Most of the time, this last step is unnecessary: it doesn't help us integrate along the curve in the slightest.

To take another example, suppose we want a parameterization of the unit circle without any sines or cosines. Well, we've seen that the portion with $y \ge 0$ has an easy parameterization in terms of x. Similarly, if we assume $y \le 0$, we can solve for x. The two parameterizations are

$$\mathbf{r}_1(t) = (t, \quad \sqrt{1-t^2}), \qquad t \in [-1,1]$$
$$\mathbf{r}_2(t) = (t, -\sqrt{1-t^2}), \qquad t \in [-1,1]$$

Using these two parameterizations together, though, *might* land us in trouble later on, because they don't have a consistent orientation. Specifically, $\mathbf{r}_1$ goes clockwise around the top of the circle, and $\mathbf{r}_2$ goes counterclockwise around the bottom of the circle. We can fix this, and go counterclockwise in both cases, by replacing $x = t$ with $x = -t$ in $\mathbf{r}_1$:

$$\begin{aligned} \mathbf{r}_1(t) &= (-t, \sqrt{1-t^2}), \qquad t \in [-1, 1] \\ \mathbf{r}_2(t) &= (t, -\sqrt{1-t^2}), \qquad t \in [-1, 1] \end{aligned}$$

(Again, we could modify $\mathbf{r}_2$ to use the interval $[1, 3]$ and combine these into a single piecewise expression, but usually we will not need to.)

3.3 Transformations of curves

A very nice feature of parameterizations is that they behave nicely with transformations. For example, suppose we want to have a circle of radius 2, rather than 1, and centered at the point $(1, 3)$, rather than $(0, 0)$. How can we accomplish this?

Well, the transformation $(x, y) \mapsto (2x, 2y)$ will keep the origin in place, but scale everything up by a factor of 2. Then, the transformation $(x, y) \mapsto (x + 1, y + 3)$ will translate $(0, 0)$ to $(1, 3)$, and everything else with it. So the transformation $(x, y) \mapsto (2x + 1, 2y + 3)$ will do one, then the other.

To apply it to the transformation, just apply this to the components of $\mathbf{r}(t)$: it's that simple! Starting from $\mathbf{r}(t) = (\cos t, \sin t)$, we can just take

$$\mathbf{r}(t) = (2\cos t + 1, 2\sin t + 3), \qquad t \in [0, 2\pi]$$

and we have a parameterization of the circle we wanted!

There are lots of transformations out there. Some of them get quite tricky without a thorough knowledge of linear algebra. We will not focus on these in detail; it's not as though we are short of tricky topics to cover.

In addition to the scaling and translation covered in the above example, it is also useful to consider the effect of permuting coordinates in a transformation. This lets us parameterize curves where the variables x, y, z play roles we're not used to.

Once again, let's begin with the unit circle. Extending our standard parameterization to $\mathbb{R}^3$, we get

$$\mathbf{r}(t) = (\cos t, \sin t, 0), \qquad t \in [0, 2\pi].$$

What if we change this parameterization to have a 0 in the y-coordinate? The result,

$$\mathbf{r}(t) = (\cos t, 0, \sin t), \qquad t \in [0, 2\pi],$$

is a parameterization of a circle of radius 1, centered at the origin, but in the xz-plane, instead.

Is this a clockwise or a counterclockwise parameterization? Does that question even make sense? We will return to that question much later in the semester.