Math 3204: Calculus IV[1]	Mikhail Lavrov

Lecture 6: Scalar line integrals

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1 Motivation for the scalar line integral

Thus far, we have dealt with integrals over a *region*. Line integrals are all the various kinds of integrals we can take over a *curve*.

1.1 Visualizing the scalar line integral

We will start with the scalar line integral, and we will start with the scalar line integral in $\mathbb{R}^2$ because it will be easier to draw diagrams that way. The inputs that feed into this are:

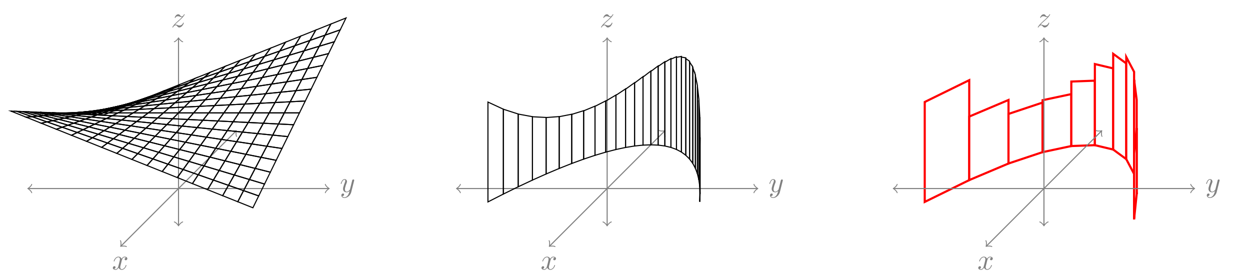
- A function of two variables, $f\colon \mathbb{R}^2 \to \mathbb{R}$.
- A curve C in $\mathbb{R}^2$.

We will write

$$\int_C f(x,y)\,\mathrm{d}s$$

for the scalar line integral of f over C. The differential element "ds" here is not the kind that has a useful meaning. You can think of the s as standing for "segment", so that ds is an infinitesimal segment of the curve C; ultimately, ds just means "we're taking a scalar line integral".

So what is this integral? Well, first, imagine that our function f is graphed in $\mathbb{R}^3$. Specifically, we plot the graph of $z = f(x,y)$, as in Figure 1a.



(a) A plot of $z = f(x,y)$ (b) The area between C and C' (c) Approximation by rectangles

Figure 1: Visualizing the scalar line integral of a 2-variable function

The 2-dimensional curve C lives in the xy-plane. We can imagine a 3-dimensional curve C' that lives on the surface $z = f(x,y)$: above every point $(x,y,0)$ on C, there is a point $(x,y,f(x,y))$ on C'. We can draw a surface in $\mathbb{R}^3$ stretched out between C and C' (as in Figure 1b). To put it

[1]This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

differently, imagine that we build a wall in the shape of C, such that at point (x, y), the height of the wall is $f(x, y)$.

The ordinary, one-dimensional integral $\int_{x=a}^{b} f(x)\, \mathrm{d}x$ represents the area between the x-axis and the graph of $y = f(x)$ over the interval $[a, b]$, at least when $f(x)$ is positive. Similarly, the scalar line integral $\int_C f(x, y)\, \mathrm{d}s$ represents the area between C and C', at least when $f(x, y)$ is positive.

As a special case, when $f(x, y) = 1$, C will be parallel to C'. In this case, $\int_C \mathrm{d}s$ will just be the length of C. This is called an **arc length** integral.

1.2 Formal definition

To define this integral formally, we go back to the intuition of Riemann sums. We can approximate the one-dimensional integral by the area of rectangles, and we can approximate the scalar line integral by the area of rectangles—just ones built in 3 dimensions. See Figure 1c for an example.

Let the curve C be parameterized by $\mathbf{r}(t)$, where $t \in [a, b]$. Divide the integral $[a, b]$ into intervals of length Δt. For each interval $[t_i, t_{i+1}]$ where $t_{i+1} = t_i + \Delta t$, we approximate a tiny portion of the area between C and C' by a rectangle. Its height is $f(\mathbf{r}(t_i))$: the vertical distance from C to C' at the point $\mathbf{r}(t)$. Its width is $\|\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)\|$: the horizontal distance between the current point $\mathbf{r}(t_i)$ and the next point $\mathbf{r}(t_{i+1})$.

Let's rewrite the area of this rectangle slightly differently:

$$f(\mathbf{r}(t_i)) \cdot \|\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)\| = f(\mathbf{r}(t_i)) \cdot \left\| \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \right\| \Delta t.$$

This makes the sum of the areas of these rectangles,

$$\sum_{i=1}^{(b-a)/\Delta t} f(\mathbf{r}(t_i)) \cdot \left\| \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \right\| \Delta t,$$

have the form of a Riemann sum. The values of some integrand at each t_i are multiplied by Δt and added up.

But what is that integrand? One of the factors is just $f(\mathbf{r}(t))$. The other factor is the norm of a difference ratio $\frac{\Delta \mathbf{r}}{\Delta t}$. When we take the limit as $\Delta t \to 0$, this ratio turns into a derivative $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$. We take the resulting integral as our definition:

$$\int_C f(x, y)\, \mathrm{d}s := \int_{t=a}^{b} f(\mathbf{r}(t)) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t.$$

Some care is necessary to prove that this is well-defined: in other words, that the result does not depend on the particular choice of parameterization $\mathbf{r}(t)$. We will not prove this, but the idea is that a different choice of parameterization corresponds to a substitution in the integral.

1.3 An example

Let's integrate $f(x, y) = x^2$ over the unit circle.

Our first step is always to parameterize the curve. Here, we will work with the standard parameterization

$$\mathbf{r}(t) = (\cos t, \sin t), \qquad t \in [0, 2\pi].$$

Our first step is to take the derivative. It is common to write $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ as $(-\sin t, \cos t)$. I will write it as $-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$, for reasons I'll explain in the next lecture. Either way, $\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\|$ is $\sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{1} = 1$ for all t.

A parameterization with this property is called a **natural parameterization** or **unit-speed parameterization** of its curve. Intuitively, if $\mathbf{r}(t)$ describes the position of a particle at time t, then $\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\|$ describes its speed at time t, which is where we get the second name.

The natural parameterization is convenient because it makes our integration easier: we just have to integrate $f(\mathbf{r}(t))$. Here, $\mathbf{r}(t) = (\cos t, \sin t)$ breaks down into $x(t) = \cos t$ and $y(t) = \sin t$, so if $f(x, y) = x^2$, then $f(\mathbf{r}(t)) = \cos^2 t$.

Therefore

$$\int_C x^2 \,\mathrm{d}s = \int_{t=0}^{2\pi} \cos^2 t \,\mathrm{d}t = \int_{t=0}^{2\pi} \frac{1 + \cos 2t}{2} \,\mathrm{d}t = \int_{t=0}^{2\pi} \frac{1}{2} \,\mathrm{d}t + \int_{t=0}^{2\pi} \frac{\cos 2t}{2} \,\mathrm{d}t = \pi + 0 = \pi.$$

2 Scalar line integrals in 3D

2.1 The definition

The scalar line integral of a function $f \colon \mathbb{R}^3 \to \mathbb{R}$ is defined in exactly the same way:

$$\int_C f(x, y) \,\mathrm{d}s := \int_{t=a}^{b} f(\mathbf{r}(t)) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t.$$

We cannot visualize the integral in exactly the same way any longer, because that would require us to plot a hypersurface $w = f(x, y, z)$ in 4-dimensional space, and that's not particularly helpful. So we need new intuition for what this integral means.

An analogy that's better than nothing (even if it's vague) is the space dust analogy. Imagine that C is the path a spaceship takes traveling through space. At every point in space, there's some amount of space dust—some regions of space may be more dusty than others, so we need a function $f(x, y, z)$ to tell us the amount of space dust at a point (x, y, z). Traveling through space, the spaceship collects all the space dust at every point it visits; the scalar line integral is the total amount of space dust collected.

If you don't like spaceships, you can tell yourself other stories about what the scalar line integral is, but it's important to remember what the scalar line integral is *not*, to make sure that your intuition does not mislead you.

For example, suppose that $f(x, y, z)$ represents the intensity of solar radiation at a point (x, y, z). A spaceship is heated up by solar radiation as it travels through space. Is it reasonable to measure the total amount of heat collected by the spaceship by a scalar line integral?

It's not! The problem is that the effect of solar radiation would also depend on time. By going faster along the same path, the spaceship can reduce the amount of solar radiation it has to endure.

On the other hand, if the spaceship stopped for a few days at a point (x, y, z), it would accumulate a few days' worth of radiation from that point. These are changes in the parameterization $\mathbf{r}(t)$ that do not change the curve C, so they *should not* affect the scalar line integral, and so that integral does not describe our solar radiation thought experiment.

(We must assume that our "space dust" is a finite resource that our spaceship collects once it visits a region, which is then used up. Traveling more slowly or more quickly will not change the amount of space dust collected.)

2.2 Examples

Let's take the function $f(x, y, z) = xyz$, and integrate it over the line segment $(0, 0, 1)$ to $(1, 1, 0)$.

We can parameterize this line segment by $\mathbf{r}(t) = (t, t, 1 - t)$ where $t \in [0, 1]$. To set up the integral, we need to find two quantities: $f(\mathbf{r}(t))$ and $\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\|$.

- $f(\mathbf{r}(t))$ is $f(x, y, z) = xyz$ evaluated at the point $(x, y, z) = (t, t, 1 - t)$: it is $t^2(1 - t)$ or $t^2 - t^3$.
- $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, so $\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$.

Integrating, we get

$$\int_C xyz\,\mathrm{d}s = \int_{t=0}^{1} (t^2 - t^3)\sqrt{3}\,\mathrm{d}t = \left(\frac{t^3}{3} - \frac{t^4}{4}\right)\sqrt{3}\Big|_{t=0}^{1} = \frac{1}{12}\sqrt{3}.$$

Let's take a different path from $(0, 0, 1)$ to $(0, 1, 0)$: the segmented path shown in Figure 2b rather than the straight line in Figure 2a that we took first.

(a) Segment from $(0, 0, 1)$ to $(1, 1, 0)$

(b) A piecewise linear path

Figure 2: Two ways to go from $(0, 0, 1)$ to $(1, 1, 0)$

Here, it makes sense to split up the integral into three integrals over the three segments C_1, C_2, and C_3. We can combine the pieces by adding up the three integrals:

$$\int_{C_1} xyz\,\mathrm{d}s + \int_{C_2} xyz\,\mathrm{d}s + \int_{C_3} xyz\,\mathrm{d}s.$$

We don't have to do anything for the first integral. Everywhere on C_1, $xyz = 0$, because $x = 0$; therefore we are integrating 0, and we get 0.

We can parameterize C_2 by $\mathbf{r}(t) = (t, 1, 1)$ where $t \in [0, 1]$. Then $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{i}$, which has norm 1, and

$$\int_{C_2} xyz \,\mathrm{d}s = \int_{t=0}^{1} (t \cdot 1 \cdot 1) \cdot 1 \,\mathrm{d}t = \frac{t^2}{2}\bigg|_{t=0}^{1} = \frac{1}{2}.$$

One way that people like thinking about such paths, where only x changes, is that we've rewritten

$$\int_{C_2} xyz \,\mathrm{d}s = \int_{x=0}^{1} x \cdot 1 \cdot 1 \,\mathrm{d}x.$$

That's because a special type of unit-speed parameterization is a parameterization where one variable is equal to t while the other two are held constant.

Similarly, for C_3, since only z changes, we can write

$$\int_{C_3} xyz \,\mathrm{d}s = \int_{z=0}^{1} 1 \cdot 1 \cdot z \,\mathrm{d}z = \frac{1}{2}.$$

(This should not be an integral from $z = 1$ to $z = 0$, or rather, we want the scalar line integral over C_3 to be unoriented.)

Altogether, the three integrals add up to $0 + \frac{1}{2} + \frac{1}{2} = 1$. It's worth pointing out that two different paths give us different answers in this setting! In general, scalar line integrals are always path-dependent: if you pick a path with more "space dust" on it, you'll collect more space dust as you travel it.

2.3 Mass of a thin wire

Scalar line integrals also come up when we approximate very thin solid regions by a curve.

For example, suppose that we have a wire bent into the parabolic curve parameterized by $\mathbf{r}(t) = (t\sqrt{2}, t\sqrt{2}, 1 - t^2)$ where $t \in [0, 1]$. The wire is not exactly shaped like the curve, but if the thickness is negligible, we might approximate it by the curve.

What if we want to find the mass of the wire?

In 3D solid regions, mass is the integral of density over the region. Here, because we're integrating over a curve, the quantity we want is not quite density. Instead, we want the **mass per unit length** of the wire at a point.

As before, this is a function δ, which might be specified in two ways. It might be a function $\delta(x, y, z)$, which tells us the mass per unit length at point (x, y, z) on the curve C. (In this case, the function might be defined even outside C, but it doesn't have meaning there.) Or, it might be a function $\delta(t)$, which tells us the mass per unit length at point $\mathbf{r}(t)$.

In this example, let's suppose that δ is specified in the second way: $\delta(t) = t$. This is a wire that's vanishingly thin at one end, and getting steadily thicker towards the other end.

The mass integral is the same as other mass integrals, except it's a line integral. We will integrate $\delta(t)$, which will take the place of $f(\mathbf{r}(t))$. We still need to work out $\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\|$: since $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} - 2t\,\mathbf{k}$, it is equal to $\sqrt{4+4t^2} = 2\sqrt{1+t^2}$. Now we can compute the mass as

$$\int_{t=a}^{b} \delta(t) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t = \int_{t=0}^{1} 2t\sqrt{1+t^2}\,\mathrm{d}t = \int_{u=1}^{2} \sqrt{u}\,\mathrm{d}u = \frac{2}{3}u^{3/2}\Big|_{u=1}^{2} = \frac{2}{3}(2\sqrt{2}-1).$$

(Here, we applied the u-substitution $u = 1+t^2$, with $\mathrm{d}u = 2t\,\mathrm{d}t$.)

In similar cases, we've also looked at the center of mass of a solid region. The center of mass of a wire can be computed in the same way.

In this problem, we would define

$$\overline{x} = \frac{\int_{t=a}^{b} x(t)\delta(t) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t}{\int_{t=a}^{b} \delta(t) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t} = \frac{\int_{t=0}^{1} (t\sqrt{2})\, 2t\sqrt{1+t^2}\,\mathrm{d}t}{\frac{2}{3}(2\sqrt{2}-1)}$$

and set up integrals for $\overline{y}$ and $\overline{z}$ in similar ways. The specific integral here is not very fun to do, so we'll skip it.