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Lecture 9: Flux and the flux integral

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1 Flux

In summary of the last few lectures, we have two line integrals that we're capable of taking right now. The first,

$$\int_C f(x, y, z)\,\mathrm{d}s,$$

is the scalar line integral, which accumulates *the value of a scalar function* f along the curve C. The second,

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s,$$

is the vector line integral, which accumulates *the amount that the vector field* $\mathbf{F}$ *follows* C along the curve C.

Both of these are equally valid in two or three dimensions; in fact, if we wanted to generalize everything we've done for these integrals to $\mathbb{R}^n$, nothing would change. Flux, the new quantity we want to introduce, is different: it will only correspond to a line integral when we're working in $\mathbb{R}^2$.

The reason is that **flux** is a measure of how much a vector field $\mathbf{F}$ crosses a boundary. Imagine stretching out a net in a body of water, or flying a kite: in both cases, you may well be interested in knowing how much water (or air) is crossing this boundary each second.

To draw a boundary between two parts of $\mathbb{R}^n$, you need an $(n-1)$-dimensional object. In $\mathbb{R}^2$, you can separate two regions by drawing a curve. In $\mathbb{R}^3$, however, it takes a surface to separate two regions. If you stretch your imagination, you might be able to picture $\mathbb{R}^4$ (4-dimensional space) and how it takes a 3-dimensional boundary there to separate two regions.

For this reason, we will only consider flux in $\mathbb{R}^2$ today, which we'll be able to describe as a line integral. We will return to flux when we begin working with surface integrals in $\mathbb{R}^3$.

In two dimensions, the flux of $\mathbf{F}$ across a curve C will measure *the amount that the vector field* $\mathbf{F}$ *crosses* C. It is a signed quantity: $\mathbf{F}$ can cross C in either direction, and we want these effects to have opposite signs.

For this reason, we need C to have an orientation, but the meaning we attach to the orientation is different. For the ordinary vector line integral, an orientation of C was a direction along C. For the flux line integral, an orientation of C is a direction *across* C. Which way of crossing C do we consider "positive", and which way do we consider "negative"?

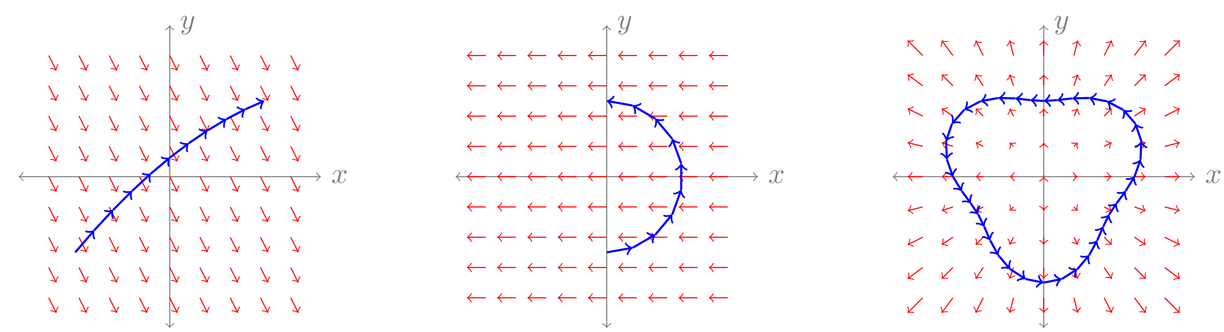
[1]This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

We will adopt a convention for turning one of these orientations into the other. This convention is arbitrary: you cannot deduce from first principles which decision we make here. The rule is:

- Our old notion of orientation for curve C in $\mathbb{R}^2$ gives a preferred "positive" direction of travel along C. As you're traveling along C in the positive direction, one side of the curve is on the left, and one side is on the right. Our convention is that **the positive direction of crossing C is from left to right**.
- Going the other way, suppose that you have a curve C for which we've picked a preferred "positive" direction of crossing C. Then we can choose a direction of travel along C so that the positive direction of crossing C is going from your left to your right. That direction of travel is going to be our positive direction along C.
- As a special case, suppose C is a simple closed curve—"simple" meaning that it does not cross itself. In this case, C is the boundary of a region R in the plane.

  By default, we prefer to measure the *outward* flux across C: that is, we want the positive direction of crossing C to be the direction leaving the region R. This means that we want the orientation along C to be chosen so that R is on our left. Therefore our default orientation of the boundary of R is the curve that goes *counterclockwise* around R.

Some examples where the flux across a curve is clearly positive or negative are shown in Figure 1. Of course, it's possible for a vector field to have a flux across a curve that's positive at some points, and negative in others. In that case, these contributions will partially cancel out when we compute the net flux across the curve.



(a) Positive flux across curve C (b) Negative flux across curve C (c) Positive outward flux

Figure 1: Three examples of positive and negative flux

2 Defining the flux integral

As motivation for the flux integral, consider the "scalar form" of the vector line integral:

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s.$$

Here, we think of the vector line integral (which, recall, computes the total amount that $\mathbf{F}$ follows the curve C) in two steps. First, we compute $\mathbf{F} \cdot \mathbf{T}$ at each point on the curve C: this measures how much $\mathbf{F}$ follows the curve C at that point. Then, we integrate that quantity over the curve C. This works out by the identity

$$\int_{t=a}^{b} (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t)) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t = \int_{t=a}^{b} \left(F(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}/\mathrm{d}t}{\|\mathrm{d}\mathbf{r}/\mathrm{d}t\|} \right) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \,\mathrm{d}t.$$

The flux integral will be defined starting from the same idea. We will write it as

$$\int_C \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}s$$

where $\mathbf{n}$ will be a unit *normal vector* to the curve C, pointing in the positive direction crossing C. (As with $\mathbf{T}$, the vector $\mathbf{n}$ is really a function $\mathbf{n}(t)$ that tells us the normal vector to the curve C at the point $\mathbf{r}(t)$.)

How do we compute $\mathbf{n}$, given $\mathbf{T}$? The idea is that the positive direction *across* a curve, from our convention, is always a 90° clockwise turn from the positive direction *along* the curve.

We will denote this clockwise turn as $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. Viewing $\mathbf{n}$ and $\mathbf{T}$ as three-dimensional vectors which happen to be parallel to the xy-plane, this $\times$ is the cross product. If you haven't dealt with the cross product before (or in a while), we'll talk about it in detail later on in the semester. For now, we just need to know that it satisfies the rule

$$(a\,\mathbf{i} + b\,\mathbf{j}) \times \mathbf{k} = b\,\mathbf{i} - a\,\mathbf{j}.$$

You can check for yourself on a couple of examples that $b\,\mathbf{i} - a\,\mathbf{j}$ is really a 90° clockwise rotation of $a\,\mathbf{i} + b\,\mathbf{j}$. (Or you can look at the examples in Figure 2.)

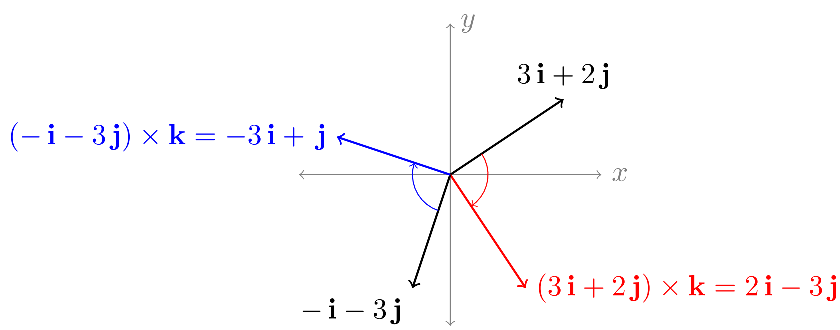


Figure 2: Rotating a vector with the $\times\mathbf{k}$ operation

Returning to our integral: the vector $\mathbf{T}$ (really, $\mathbf{T}(t)$) is a scaled version of the vector $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, and the operations of rotating a vector and scaling a vector commute. So instead of rotating $\mathbf{T}$ by 90°, we can rotate $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ by 90°. This gives us a formula for the flux integral:

$$\int_C \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}s = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{k} \right) \mathrm{d}t.$$

Before we do an example, there is another way to look at the flux integral. If $\mathbf{r}(t) = (x(t), y(t))$, then $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathbf{j}$, which means that $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times k = \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathbf{i} - \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathbf{j}$. Now let's write $\mathbf{F}$ as $M\,\mathbf{i} + N\,\mathbf{j}$ and expand out the integrand.

We get:

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{k}\right) &= (M(\mathbf{r}(t))\,\mathbf{i} + N(\mathbf{r}(t))\,\mathbf{j}) \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathbf{i} - \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathbf{j}\right) \\ &= M(\mathbf{r}(t))\frac{\mathrm{d}y}{\mathrm{d}t} - N(\mathbf{r}(t))\frac{\mathrm{d}y}{\mathrm{d}x}.\end{aligned}$$

So it makes sense to write the flux integral as

$$\int_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s = \int_{t=a}^{b} \left(M(\mathbf{r}(t))\frac{\mathrm{d}y}{\mathrm{d}t} - N(\mathbf{r}(t))\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \int_C M\,\mathrm{d}y - N\,\mathrm{d}x,$$

in a similar way to how we can write the vector line integral as

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C M\,\mathrm{d}x + N\,\mathrm{d}y.$$

From this point of view, the integral of a differential 1-form like, say, $e^{x+y}\,\mathrm{d}x + \cos y\,\mathrm{d}y$ around C is indifferent to how we arrived at it. We can think of it as the integral of $\mathbf{F}_1 = e^{x+y}\,\mathbf{i} + \cos y\,\mathbf{j}$ along C, or we can think of it as the flux integral of $\mathbf{F}_2 = \cos y\,\mathbf{i} - e^{x+y}\,\mathbf{j}$ across C: either way, we would arrive at the integral

$$\int_C e^{x+y}\,\mathrm{d}x + \cos y\,\mathrm{d}y$$

which we would compute in the same way.

3 Examples of flux integrals

First, let's do a quick example which we can expect to be 0. Take $\mathbf{F}$ to be the constant vector field $2\,\mathbf{i} + 3\,\mathbf{j}$, and take C to be the counterclockwise unit circle. The flux of $\mathbf{F}$ across C is the flux of $\mathbf{F}$ *outward* from the unit circle... but we expect this to be 0, because on net, $\mathbf{F}$ is neither entering nor leaving the unit circle. The amount that $\mathbf{F}$ enters the circle from one side is exactly equal to the amount that $\mathbf{F}$ leaves the circle from the other side. (See Figure 3a.)

(a) Outward flux of the vector field $\mathbf{F} = 2\,\mathbf{i} + 3\,\mathbf{j}$ across the unit circle

(b) Outward flux of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ across the boundary of a triangle

Figure 3: Two flux integrals

We can verify that this intuition is correct by computing the flux integral.

If we parameterize the unit circle by $\mathbf{r}(t) = (\cos t, \sin t)$, then $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$, and therefore $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{k} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$. In other words, $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{k}$ is equal to the radial vector pointing from the origin to $\mathbf{r}(t)$. This makes sense: the normal vector pointing out of the circle is the vector pointing directly away from $(0,0)$, the center of the circle.

The vector field $\mathbf{F}$ is $2\,\mathbf{i} + 3\,\mathbf{j}$, which we don't even need to evaluate at $\mathbf{r}(t)$, because it's constant. Therefore $\mathbf{F}(\mathbf{r}(t)) \cdot \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{k}\right)$ is equal to $2\cos t + 3\sin t$, and the flux integral is equal to

$$\int_{t=0}^{2\pi} (2\cos t + 3\sin t)\,\mathrm{d}t = (2\sin t - 3\cos t)\Big|_{t=0}^{2\pi} = 0.$$

By the way, there's a trend you should grow used to seeing in these integrals: when you integrate $\sin t$ or $\cos t$ over an entire period of that function, you get 0.

Now let's do another integral. Here, our vector field will be the expanding vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$. Our curve will be the boundary of the triangle with corners at $(1,-1)$, $(0,1)$, and $(-1,-1)$. We want the outward flux across this boundary, so the curve should be oriented counterclockwise.

The triangle has a piecewise boundary, so we'll compute the flux integral piece by piece, and we'll see three ideas as we go.

First, the piece from $(1,-1)$ to $(0,1)$. This is parameterized by $\mathbf{r}(t) = (1-t, 2t-1)$, where $t \in [0,1]$. This means that $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -\mathbf{i} + 2\,\mathbf{j}$, and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{k} = 2\,\mathbf{i} + \mathbf{j}$. Meanwhile, $\mathbf{F}(\mathbf{r}(t)) = (1-t)\,\mathbf{i} + (2t-1)\,\mathbf{j}$, so the flux integral across this side of the triangle is

$$\int_{t=0}^{1} ((1-t)\,\mathbf{i} + (2t-1)\,\mathbf{j}) \cdot (2\,\mathbf{i} + \mathbf{j})\,\mathrm{d}t = \int_{t=0}^{1} (2 - 2t + 2t - 1)\,\mathrm{d}t = \int_{t=0}^{1} 1\,\mathrm{d}t = 1.$$

Second, the piece from $(0,1)$ to $(-1,-1)$. Here, we'll use the other expression for the flux integral: it is

$$\int_C M\,\mathrm{d}y - N\,\mathrm{d}x = \int_C x\,\mathrm{d}y - y\,\mathrm{d}x = \int_{t=0}^{1} \left(x(t)\,\frac{\mathrm{d}y}{\mathrm{d}t} - y(t)\,\frac{\mathrm{d}x}{\mathrm{d}t}\right)\,\mathrm{d}t.$$

The parameterization of this line segment is $\mathbf{r}(t) = (-t, 1-2t)$, where $t \in [0,1]$. Therefore $x(t) = -t$, $y(t) = 1 - 2t$, $\frac{\mathrm{d}x}{\mathrm{d}t} = -1$, and $\frac{\mathrm{d}y}{\mathrm{d}t} = -2$, so we get

$$\int_{t=0}^{1} (-t(-2) - (1-2t)(-1))\,\mathrm{d}t = \int_{t=0}^{1} 1\,\mathrm{d}t = 1.$$

It's not surprising that this is identical to the previous integral: these two sides of the triangle are symmetric.

For the final piece, from $(-1,-1)$ to $(1,-1)$, we'll think of the flux integral in its "scalar integral version": as

$$\int_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s.$$

We can look at the boundary and directly see what $\mathbf{n}$ should be: it is the unit normal vector pointing out of this boundary of the circle, and so it should point straight down: $\mathbf{n} = -\mathbf{j}$. (In general, $\mathbf{n}$ is a function of t, but here it is a constant function.) Therefore $\mathbf{F} \cdot \mathbf{n} = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-\mathbf{j}) = -y$.

Moreover, $-y$ is the constant 1 over this entire piece, and the scalar line integral of 1 over any curve is just its arc length. The length of this side of the triangle is 2, so the flux across it is 2. We get a total flux of $1 + 1 + 2 = 4$ out of the triangle.