Calculus IV Homework 2

Mikhail Lavrov

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1. Find a parameterization for each of the following curves.
   (a) The curve in $\mathbb{R}^2$ that traces the polar equation $r = \sin 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.
   (b) The curve in $\mathbb{R}^3$ that lies on the surface of a unit sphere centered at the origin, and goes from the top $(0, 0, 1)$ to the bottom $(0, 0, -1)$ while staying at a constant polar angle of θ_0. (Express your answer in terms of θ_0.)
   (c) The curve in $\mathbb{R}^3$ that follows the path of a particle traveling in the plane $z = x + y + 2$ whose shadow in the xy-plane follows the unit circle.
2. For each of the following oriented curves, split it up into pieces and parameterize each piece, taking care that your parameterizations respect the orientation of the curve.
   (a) The curve in $\mathbb{R}^2$ that goes counterclockwise around the triangle with vertices $(0, 0)$, $(3, 1)$, and $(2, 3)$.
   (b) The curve in $\mathbb{R}^2$ that goes from $(-1, 1)$ to $(2, 4)$ by following the parabola $y = x^2$, then returns along a straight line.
   (c) The curve in $\mathbb{R}^3$ that goes from $(2, 0, 0)$ to $(-2, 0, 0)$ along a half-circle in the xy-plane, and returns along a half-circle in the xz-plane.

   Both half-circles should be centered at the origin; the first should stay in the $y \geq 0$ half of the xy-plane, and the second should stay in the $z \geq 0$ half of the xz-plane.
3. Integrate the scalar function $f(x, y, z) = \frac{x}{\sqrt{1+2y}}$ along the curve in $\mathbb{R}^3$ which goes from $(0, 0, 0)$ to $(1, 1, \frac{2}{3})$ parameterized by $\mathbf{r}(t) = (t, t^2, \frac{2}{3}t^3)$, where $t \in [0, 1]$.
4. A copper wire of uniform thickness is first bent into the shape of a circle of radius 1, then cut into three equal pieces. One piece (that is, one 120° arc of the circle) is placed in the xy-plane so that one endpoint is at $(1, 0)$ and the wire follows the unit circle counterclockwise. The other two pieces are discarded.

   Find the coordinates of the center of mass of this piece of wire.
5. A Bézier curve is a curve specified by a set of "control points", often used in computer graphics. It is a generalization of our parameterization of a line segment from point $\mathbf{a}$ to point $\mathbf{b}$, as follows:

- A Bézier curve with two control points $\mathbf{a}_1$, $\mathbf{a}_2$ is just a line segment from $\mathbf{a}_1$ to $\mathbf{a}_2$, parameterized by $\mathbf{r}(t) = (1-t)\mathbf{a}_1 + t\mathbf{a}_2$, where $t \in [0,1]$.
- To define a Bézier curve with three control points $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, first let $\mathbf{r}_{12}(t)$ be the parameterization of the line segment from $\mathbf{a}_1$ to $\mathbf{a}_2$ (as above), and let $\mathbf{r}_{23}(t)$ be the same kind of parameterization of the line segment from $\mathbf{a}_2$ to $\mathbf{a}_3$. Then the Bézier curve we want is

  $$\mathbf{r}(t) = (1-t)\mathbf{r}_{12}(t) + t\mathbf{r}_{23}(t), \qquad t \in [0,1].$$

  In other words, $\mathbf{r}(t)$ is a point on a moving line segment which starts on the segment parameterized by $\mathbf{r}_{12}(t)$ and ends on the segment parameterized by $\mathbf{r}_{23}(t)$.

In the same way, we can define a Bézier curve with four control points in terms of two Bézier curves with just three control points, and so forth, but we're not going to go that far.

Instead:

(a) Find the parameterization $\mathbf{r}(t)$ of the Bézier curve with control points $(-1,0)$, $(1,1)$, and $(0,-1)$.

(b) Find the arc length of the Bézier curve you found in part (a).