# Calculus IV Homework 7

Mikhail Lavrov

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1. Find the curl of the following vector fields:
   (a) $\mathbf{F} = \cos(e^x)\,\mathbf{i} + \sqrt{y^3 + y^6 + y^7}\,\mathbf{j} + z^{1+\tan^2 z}\,\mathbf{k}$.
   (b) $\mathbf{F} = yz\,\mathbf{i} - 2xz\,\mathbf{j} + xy\,\mathbf{k}$.
   (c) $\mathbf{F} = xy\cos z\,\mathbf{i} + xy\cos z\,\mathbf{j} + xy\cos z\,\mathbf{k}$.
2. Each part of this problem gives you a surface $S$ with a parameterization. Use that surface parameterization to find four parameterizations $\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t), \mathbf{r}_4(t)$ for the boundary of $S$. Then, list which (If any) of these boundaries cancel with each other, and which (if any) are trivial; briefly describe the remaining boundaries geometrically.

(example) The cylinder parameterized by $\mathbf{r}(u, v) = (\cos u, \sin u, v)$, where $(u, v) \in [0, 2\pi] \times [0, 3]$.

*Solution: the four parameterizations we get for the boundary are are*

$$\begin{aligned}
\mathbf{r}_1(t) &= \mathbf{r}(t, 0) = (\cos t, \sin t, 0) && t \in [0, 2\pi] \\
\mathbf{r}_2(t) &= \mathbf{r}(2\pi, t) = (1, 0, t) && t \in [0, 3] \\
\mathbf{r}_3(t) &= \mathbf{r}(-t, 3) = (\cos t, -\sin t, 3) && t \in [-2\pi, 0] \\
\mathbf{r}_4(t) &= \mathbf{r}(0, -t) = (1, 0, -t) && t \in [-3, 0]
\end{aligned}$$

*Here, $\mathbf{r}_1(t)$ and $\mathbf{r}_3(t)$ are circles around the top and bottom edges of the cylinder, and $\mathbf{r}_2(t)$ and $\mathbf{r}_4(t)$ cancel with each other.*

   (a) The portion of the unit sphere with $x \geq 0$, $y \geq 0$, and $z \geq 0$, which is parameterized by $\mathbf{r}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$, where $(u, v) \in [0, \pi/2] \times [0, \pi/2]$.
   (b) The portion of the paraboloid $x = y^2 + z^2$ with $0 \leq x \leq 1$, which is parameterized by $\mathbf{r}(u, v) = (u^2, u\cos v, u\sin v)$, where $(u, v) \in [0, 1] \times [0, 2\pi]$.
3. Let $S$ be the portion of the cone with equation $z^2 = x^2 + y^2$ which lies between the planes $z = 1$ and $z = 2$, oriented so that the normal vectors point inward and upward.
   (a) Give $S$ a parameterization of the form $\mathbf{r}(u, v)$, where $(u, v) \in [a, b] \times [c, d]$, making sure to be consistent with the orientation of $S$ we want.
   (b) As you did in problem 2, use this parameterization to find the boundaries of $S$.

(c) Let $\mathbf{F}$ be the vector field $xy\,\mathbf{i} + z^2\,\mathbf{j} - (x-z)\,\mathbf{k}$. Stokes' theorem relates the curl integral of $\mathbf{F}$ over $S$ to the circulation integral of $\mathbf{F}$ along the boundary of $S$. The surface in this problem has two boundaries which we can call $C$ and $C'$. This means that in this case, Stokes' theorem tells us that

$$\iint_S \_\_\_\_\_ \,\mathrm{d}y \wedge \mathrm{d}z + \_\_\_\_\_ \,\mathrm{d}z \wedge \mathrm{d}x + \_\_\_\_\_ \,\mathrm{d}x \wedge \mathrm{d}y = \int_C \_\_\_\_\_ \,\mathrm{d}x + \_\_\_\_\_ \,\mathrm{d}y + \_\_\_\_\_ \,\mathrm{d}z$$
$$+ \int_{C'} \_\_\_\_\_ \,\mathrm{d}x + \_\_\_\_\_ \,\mathrm{d}y + \_\_\_\_\_ \,\mathrm{d}z.$$

Fill in the blanks, and describe $C$ and $C'$ geometrically. You do not have to evaluate the integrals.

4. Let $S$ be the rectangle in the plane $y+z=1$ which has corners at $(0,0,1)$, $(1,0,1)$, $(1,1,0)$, and $(0,1,0)$.

   Find the curl integral of $\mathbf{F} = xyz\,\mathbf{i} - xz\,\mathbf{j} + y^2\,\mathbf{k}$ over $S$.

5. The cone $z^2 = x^2 + y^2$ and the plane $x = 2z - 3$ intersect in an ellipse $C$ which has parameterization $\mathbf{r}(t) = (1 + 2\cos t, \sqrt{3}\sin t, 2 + \cos t)$, $t \in [0, 2\pi]$.

   (a) Find the circulation of $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$ around $C$ (using the parameterization above). We covered circulation integrals much earlier this semester, but you should still remember how to do them!

   (b) The portion of the cone above the $xy$-plane but below the plane $x = 2z - 3$ is a surface with boundary $C$. One way to take a flux integral across this surface is to treat it as a surface defined by the implicit equation $x^2 + y^2 - z^2 = 0$ above a region $R$ in the $xy$-plane.[1]

   Write down a curl integral over $S$ which is equal to the circulation of $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$ around $C$, and transform it into a double integral over $R$ with respect to $x$ and $y$. You do not have to simply it further after that (and you should feel free to leave it in $\iint_R$ form).

[1] More precisely, $R$ happens to be the interior of another ellipse: it is the region defined by $3(x-1)^2 + 4y^2 \le 12$.