Calculus IV Homework 8

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due Friday, December 1, 2023

0 Course evaluations

Course evaluations for this class can now be filled out!! There should be a link, I believe, from the D2L home page. (Let me know if you have trouble finding them.)

The *last* day that you can fill out your course evaluations is Monday, December 4^{th}.

If more than 50% of the class fills out the course evaluation, then I will wear a funny hat for the final exam.

1 The divergence theorem

1. Find the outward flux of $\mathbf{F} = (x - 2y)\,\mathbf{i} - (3z + 4x)\,\mathbf{j} + (5y + 6z)\,\mathbf{k}$ across the surface of the cube of side length 2 in $\mathbb{R}^3$ given by the inequalities $|x| \le 1$, $|y| \le 1$, and $|z| \le 1$.

2. Let S_1 be the portion of the unit sphere with $z \ge 0$, oriented with normal vector pointing up and out of the sphere.

   Let S_2 be the portion of the unit sphere with $z \le 0$, oriented with normal vector pointing up and into the sphere.

   Finally, let $\mathbf{F} = 2y\,\mathbf{i} + 2z\,\mathbf{j} + 2x\,\mathbf{k}$.

   (a) Find a vector field $\mathbf{G}$ such that the curl $\nabla \times \mathbf{G}$ is equal to $\mathbf{F}$. *(There are infinitely many possibilities, but I've set the problem up so that some are particularly easy to find.)*

   (b) Use Stokes' theorem to explain why the flux integral of $\mathbf{F}$ across S_1 is equal to the flux integral of $\mathbf{F}$ across S_2.

   (c) Use the divergence theorem to explain why the flux integral of $\mathbf{F}$ across S_1 is equal to the flux integral of $\mathbf{F}$ across S_2.

3. Let D be the cone of height 1 above the unit circle in the xy-plane: the cone described by

   $$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1 \text{ and } 0 \le z \le 1 - \sqrt{x^2 + y^2}.\}$$

   Let $\mathbf{F} = (x + y)^2\,\mathbf{i} + z^2\,\mathbf{k}$.

(a) The cone D has two boundaries: a flat disk in the xy-plane, and a lateral boundary that lies on the graph of $z = 1 - \sqrt{x^2 + y^2}$.

Explain why the (outward) flux integral of $\mathbf{F}$ across the boundary in the xy-plane is 0.

(b) Use the divergence theorem to find the outward flux integral of $\mathbf{F}$ across the lateral boundary of the cone.

2 Review

4. Let $R = \{(x, y) \in \mathbb{R}^2 : x^2 + 2xy + 2y^2 \leq 1\}$. Let S be the surface in $\mathbb{R}^3$ that we get by thinking of R as a surface in the plane $z = 0$, oriented with normal vector pointing upward.

   (a) If $\mathbf{F} = x^2\,\mathbf{k}$, explain why the following two quantities are equal:

$$\iint_R x^2 \,\mathrm{d}A = \iint_S \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}S.$$

   (b) Evaluate $\iint_R x^2 \,\mathrm{d}A$ by making the uv-substitution $x = u\cos v - u\sin v$ and $y = u\sin v$.

   (c) Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}S$ by giving S the parameterization

$$\mathbf{r}(u, v) = (u\cos v - u\sin v, u\sin v, 0), \qquad (u, v) \in [0, 1] \times [0, 2\pi].$$

5. Parameterize the following objects:

   (a) The parallelogram with corners at $(0, 0, 0)$, $(0, 1, 1)$, $(2, 2, 1)$, and $(2, 1, 0)$.

   (Find a paramerization with rectangular domain $(u, v) \in [a, b] \times [c, d]$.)

   (b) A "washer" shape in the plane $y = 1$ bounded by the inequalities $1 \leq x^2 + z^2 \leq 4$.

   (Find a paramerization with rectangular domain $(u, v) \in [a, b] \times [c, d]$.)

   (c) The curve where the cylinder $x^2 + y^2 = 1$ intersects the plane $x + y + z = 1$.

   (This is a curve, so find a parameterization with one variable $t \in [a, b]$.)