Graph Theory Homework 1

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due Friday, August 25, 2023

1 Short answer

In this section, all I expect you to do is find the answer, possibly accompanied by a diagram or a one-sentence explanation if I ask for it.

1. Imagine a puzzle in which four coins are lined up in a row: two pennies and two dimes. In one step, you can swap two adjacent coins.

   (a) Let G be the graph whose vertices are possible states of this puzzle, with an edge between states that are one step apart. *(Do not include "steps" that don't do anything because they swap two identical coins.)*

   Draw a diagram of G.

   (b) What is the order of G: the number of vertices?

   What is the size of G: the number of edges?

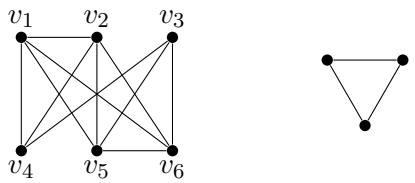
   (c) Find a cycle in G.

2. Let H be the graph whose vertices are the numbers $1, \ldots, 15$, with an edge between a and b if $|a - b| = 4$ or if $|a - b| = 10$. (For example, vertex 11 is adjacent to vertices 7 and 15, because $|11 - 7| = |11 - 15| = 4$, as well as to vertex 1, because $|11 - 1| = 10$.

   (a) Draw a diagram of H.

   (b) What are the connected components of H?

3. Let K be the graph shown below on the left. Inside K, we want to find subgraphs that look like the graph shown below on the right: subgraphs that have 3 vertices and 3 edges.



   How many such subgraphs are there? List them all, by telling me which vertices are part of each subgraph.

4. Find a connected 8-vertex graph which has K_4 as a subgraph, and has diameter 5. Point out the subgraph, and the two vertices at distance 5 from each other.

2 Proof

In this section, you should write a proof. Write in complete sentences and justify your logic. I am not grading the length of your proof, only its correctness, but a typical solution can be a paragraph long.

5. The **crown graph** on $2n$ vertices is defined to be the following bipartite graph: it has vertices $\{x_1, x_2, \ldots, x_n\}$ on one side, vertices $\{y_1, y_2, \ldots, y_n\}$ on the other side, and an edge $x_i y_j$ whenever $i \neq j$.

   Prove that, for all $n \geq 3$, this graph is connected and has diameter 3.

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 2.