Graph Theory Homework 2

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1 Short answer

1. Define Q_n, as in class, to be the graph with vertex set $\{0,1\}^n$ and an edge between any two vertices that differ in one position. Let G_n be the subgraph of Q_n induced by the vertices in which the total number of 1's is either one or two.

   (a) Draw a diagram of G_4. (It has 10 vertices: $0001, 0010, 0100, 1000$ with one 1, and $0011, 0101, 0110, 1001, 1010, 1100$ with two 1s.)

   (b) Determine the degree sequence of G_n (in general, as a function of n). Explain which vertices have which degrees.

2. Suppose that H is a bipartite graph. On one side of the bipartition, there are n vertices; their degrees are $1, 2, 3, \dots, n$. On the other side of the bipartition, there are also n vertices; all of them have degree 4. What is n?

3. What is the maximum number of edges in an 8-vertex graph with minimum degree 2 and maximum degree 3?

   Give an example of a graph with this number of edges, and a brief explanation for why that number cannot be larger.

2 Proof

4. The **crown graph** on $2n$ vertices is defined to be the following bipartite graph: it has vertices $\{x_1, x_2, \dots, x_n\}$ on one side, vertices $\{y_1, y_2, \dots, y_n\}$ on the other side, and an edge $x_i y_j$ whenever $i \neq j$.

   Prove that, for all $n \geq 3$, this graph is connected and has diameter 3.

   You have already written a rough draft of the solution; now, write a final draft.

5. Prove the following by induction on n. For all $n \geq 5$, there exists a graph with n vertices and $2n - 4$ edges that has minimum degree 2 and maximum degree 4.

   (If you're having trouble, begin by omitting the "maximum degree 4" condition; then, see if you can find a way to modify your proof to include it.)

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 3.