Math 3324: Enumerative Combinatorics[1] Mikhail Lavrov

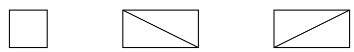
Lecture 20: Catalan Numbers

March 28, 2024 *Kennesaw State University*

1 A tiling example

Before we go on to Catalan numbers, the topic of today's lecture, let's solve a tiling problem using the idea of convolutions.

Suppose that you are laying out tiles in a long line of height 1 (just as in our Fibonacci tiling problem). This time, however, you have three tiles available to you:



That is, we have a single 1×1 tile available to use, but two different 1×2 tiles with different patterns, and it matters which 1×2 tile we use. Let's find the generating function where the coefficient of x^n is the number of ways to tile a $1 \times n$ rectangle with these tiles.

The main difference from the problems we looked at in the previous lecture is that we are not combining a fixed number of tiles: we have any number of tiles to use. This will involve a new step in our solution, so let's wait to figure out what it is and ask a simpler question. Suppose I tell you that I want to use exactly 5 tiles in total. What is the generating function now?

To solve this, we begin with the generating function for a single tile: $x + 2x^2$. Here, the coefficient of x is 1 because there is 1 way to tile a 1×1 rectangle with a single tile: use the square tile. The coefficient of x^2 is 2 because there are 2 ways to tile a 1×2 rectangle with a single tile: we can use either of the long tiles.

To obtain all the possible ways to combine 5 tiles, we can take the 5^{th} power of this generating function:

$$(x + 2x^2)^5 = x^5 + 10x^6 + 40x^7 + 80x^8 + 80x^9 + 32x^{10}.$$

This tells us that, for example, there are 10 ways to tile a 1×6 rectangle with 5 of our tiles. (What are those 10 ways? This case is not hard to check by hand.)

Now let's return to the original question: suppose we want to tile a $1 \times n$ rectangle, and we don't care about the number of tiles. What do we do now?

As a general pattern in enumerative combinatorics, you should begin to develop a reflex: whenever you see "we don't care about the value of k", you should think "sum over all possible values of k". This is how we combine many cases. So if we don't care about the number of tiles, we should sum over all the possible numbers of tiles we can use.

If we use k tiles, the generating function we get is $(x + 2x^2)^k$, generalizing our previous solution. The value of k can be any nonnegative integer: we'll allow $k = 0$ because it will make our sum look

[1]This document comes from the Math 3324 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3324-spring-2024.php

nicer without affecting things much. (With 0 tiles, we can only tile a 1×0 "rectangle".) This gives us a final answer of

$$\sum_{k=0}^{\infty}(x+2x^2)^k = \frac{1}{1-x-2x^2}$$

where, in the final step, we used the formula for an infinite geometric series that starts at 1 and has common ratio $x + 2x^2$.

We can get a closed form for this sequence in the same way that we got a closed form for the Fibonacci numbers. Begin with a partial fraction decomposition:

$$\frac{1}{1-x-2x^2} = \frac{1/3}{1+x} + \frac{2/3}{1-2x}.$$

Now, we can use the formula for an infinite geometric series in reverse on each term:

$$\frac{1}{1-x-2x^2} = \frac{1}{3}\sum_{k=0}^{\infty}(-x)^k + \frac{2}{3}\sum_{k=0}^{\infty}(2x)^k = \sum_{k=0}^{\infty}\frac{(-1)^k + 2\cdot 2^k}{3}\cdot x^k.$$

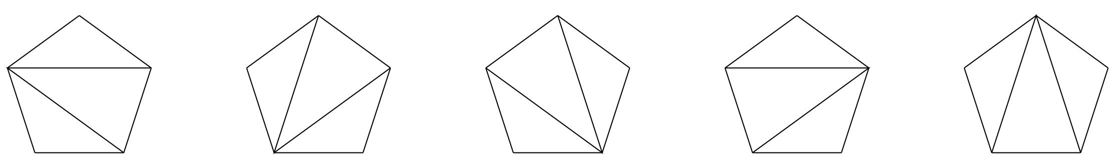
Altogether, there are $(2^{k+1} + (-1)^k)/3$ ways to tile a $1 \times n$ rectangle with our tiles!

2 The Catalan numbers

2.1 Combinatorial interpretations

Take a look at the following problems:

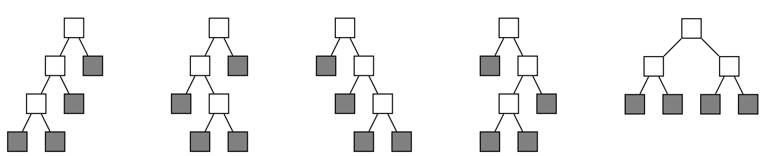
Problem 1. We have a regular n-gon, and we want to triangulate it: to draw in some of its diagonals and divide it up into $n-2$ triangles. How many ways are there to do this? Here are the solutions for $n = 5$:



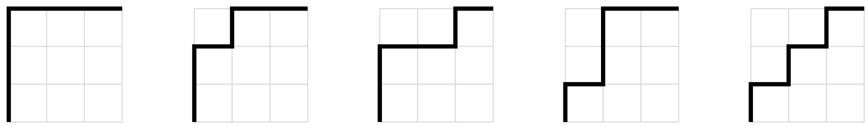
Problem 2. We want to write a sequence of n opening and n closing parentheses that match up into pairs; for example, "())(" is not valid, because the second ")" does not close a "(". Here are the solutions for $n = 3$:

()()()   ()(())   (())()   (()())   ((()))

Problem 3. We want to draw a "full binary tree" with n leaves. It will have $n-1$ non-leaf nodes, and each of those should have a left child and a right child. Here are the solutions for $n = 4$ (with leaves shaded in for clarity):



Problem 4. We want to draw a path from in an $n \times n$ grid that starts at the bottom left corner, ends at the top right corner, and moves up or to the right at each step. We've solved this problem before, but now we add an extra condition: we must never go below the bottom-left-to-top-right diagonal. Here are the solutions for $n = 3$:



All four problems have the same number of solutions! (Up to a shift in indices: for example, as seen above, triangulations of a 5-sided polygon in Problem 1 correspond to sequences with 3 pairs of parentheses in Problem 2, trees with 4 leaves in Problem 3, and paths in a 3×3 grid in Problem 4.) That number of solutions is given by the Catalan sequence $C_0, C_1, C_2, \dots$. The convention for which Catalan number is the n^{th} Catalan number matches the description in Problem 2 or Problem 4. With this convention, the sequence begins:

$$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14,\ C_5 = 42,\ C_6 = 132,\ \dots$$

2.2 Obtaining the generating function

To find a generating function and eventually a formula for the Catalan numbers, let's look at Problem 3, where we want C_n to be the number of full binary trees with $n+1$ leaves and n non-leaf nodes. I've chosen this problem because it's the one in which an important feature of the Catalan numbers becomes the most obvious: we solve the size-n problem by putting together solutions to two smaller problems. In Problem 3, this means we put together two smaller full binary trees by suspending both of them from a common root. See if you can identify this breakdown into two smaller problems in Problems 1, 2, and 4.

Our big tree has n non-leaf nodes. What can we say about the left subtree and the right subtree? Each of them has between 0 and $n-1$ non-leaf nodes, and together, they have $n-1$ non-leaf nodes in total: the n^{th} non-leaf node is always the root of the big tree, which is not part of either subtree. Let k be the number of non-leaf nodes in the left subtree—so that $n-k-1$ is the number of non-leaf nodes in the right subtree. Then we can combine C_k possible left subtrees and C_{n-k-1} possible right subtrees to get our big tree. This can be done in $C_k C_{n-k-1}$ ways, and we want to sum over all possible values of k, giving us the following recurrence relation:

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} = C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-2} C_1 + C_{n-1} C_0.$$

We can check that the first few terms of the sequence, written above, satisfy this recurrence relation. For example,

$$42 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1.$$

What does this recurrence relation look like? It's a convolution... of the Catalan numbers with themselves! To state this formally, let's define a generating function:

$$C(x) = C_0 + C_1 x + C_2 x_2 + C_3 x^3 + C_4 x^4 + \ldots = 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots$$

To get a convolution of the Catalan sequence with itself, we want to square this generating function. In $C(x)^2$, we obtain an x^n term whenever we multiply $C_k x^k$ from the first factor by $C_{n-k}x^{n-k}$ from the second factor, for any value of k. As a result, in $C(x)^2$, the coefficient of x^n is

$$C_0C_n + C_1C_{n-1}C_2C_{n-2} + \cdots + C_{n-1}C_1 + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}.$$

Our recurrence relation says that this should be equal to C_{n+1}. Therefore

$$C(x)^2 = \sum_{n=0}^{\infty} C_{n+1}x^n.$$

This sum is a left shift of $C(x)$: it is equal to $\frac{C(x)-C_0}{x}$, or $\frac{C(x)-1}{x}$. Therefore

$$C(x)^2 = \frac{C(x)-1)}{x} \iff xC(x)^2 = C(x) - 1.$$

We are almost ready to write down a generating function. To do so, we need to solve for $C(x)$. Unlike every other example we've done, though, this example is a *quadratic* equation for $C(x)$. To solve it, we'll need to use the quadratic formula on the equation $xC(x)^2 - C(x) + 1 = 0$. With $a = x$, $b = -1$, and $c = 1$, we get

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

The $\pm$ is a bit of a concern: shouldn't we just have *one* generating function? Yes, we should—and the only valid operation to put there is minus ($-$). This is not immediately obvious! It will be easier to see after we do some more work to figure out how to write $C(x)$ as a power series, and extract a formula for C_n.

2.3 A formula for the Catalan numbers

We can start by looking at $\sqrt{1-4x}$. This is $(1-4x)^{1/2}$, and we can use the binomial theorem to write down the following series for it:

$$\sum_{n=0}^{\infty} \binom{1/2}{n}(-4x)^n = 1 + \binom{1/2}{1}(-4x)^1 + \binom{1/2}{2}(-4x)^2 + \binom{1/2}{3}(-4x)^3 + \binom{1/2}{4}(-4x)^4 + \dots$$

Is this okay? It is, with the proper interpretation of $\binom{1/2}{n}$.

The formula $\binom{n}{k} = \frac{(n)_k}{k!}$ is something we originally intended for integer inputs. But we can use it for evaluating something like $\binom{1/2}{4}$ as well:

$$\binom{1/2}{4} = \frac{(1/2)_4}{4!} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!} = \frac{15/16}{24} = \frac{5}{128}.$$

It's a bit harder to describe what will happen for $\binom{1/2}{n}$ in general. The same reasoning tells us

$$\binom{1/2}{n} = \frac{(1/2)_n}{n!} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!} = (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n \cdot n!}.$$

We're fine with $n!$ in a formula; we'd rather not deal with the product of odd numbers as well. To handle it, let's multiply and divide by $2 \cdot 4 \cdot 6 \cdots (2n-2)$, which is $2^{n-1}(n-1)!$:

$$\binom{1/2}{n} = (-1)^{n-1}\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n \cdot n!} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)}{2^{n-1} \cdot (n-1)!} = (-1)^{n-1}\frac{(2n-2)!}{2^{2n-1} \cdot n! \cdot (n-1)!}.$$

(This is only valid for $n \geq 1$; we can't make sense of $(2n-2)!$ when $n = 0$. But for $n = 0$, we know that $\binom{1/2}{n}$ must be 1, because $\binom{x}{0} = 1$ for any x.)

As a result, we get

$$\sqrt{1-4x} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n = 1 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-2)!}{2^{2n-1} \cdot n! \cdot (n-1)!}(-4x)^n.$$

This is as bad as the expression will get: now, a lot of things cancel. The $(-1)^{n-1}$ and $(-4)^n$ turn into -4^n, and when we divide by 2^{2n-1}, we're left with just -2. Therefore

$$\sqrt{1-4x} = 1 + \sum_{n=1}^{\infty}\frac{-2(2n-2)!}{n!(n-1)!}x^n = 1 - \frac{2 \cdot 0!}{1!\,0!}x - \frac{2 \cdot 2!}{2!\,1!}x^2 - \frac{2 \cdot 4!}{3!\,2!}x^3 - \frac{2 \cdot 6!}{4!\,3!}x^4 - \dots.$$

Looking at this, we can see *why* we should consider $1 - \sqrt{1-4x}$ in our sum, and not $1 + \sqrt{1-4x}$. That's the option that will give us positive terms in our generating function! It will also give us a 0 constant term, which is important for our final step of dividing by $2x$. When we do so, we'll get

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{\frac{2 \cdot 0!}{1!\,0!}x + \frac{2 \cdot 2!}{2!\,1!}x^2 + \frac{2 \cdot 4!}{3!\,2!}x^3 + \frac{2 \cdot 6!}{4!\,3!}x^4 + \dots}{2x}$$

$$= \frac{0!}{1!\,0!} + \frac{2!}{2!\,1!}x + \frac{4!}{3!\,2!}x^2 + \frac{6!}{4!\,3!}x^3 + \frac{8!}{5!\,4!}x^4 + \dots$$

To find a formula for C_n, we take the coefficient of x^n, which we see is $\frac{(2n)!}{(n+1)!\,n!}$. There's a more intuitive way to write this: C_n is *very nearly* a binomial coefficient, except that $n+1$ and n don't quite add up to $2n$. We can fix this by factoring out a $\frac{1}{n+1}$ to change $(n+1)!$ to $n!$, or by multiplying by $\frac{2n+1}{2n+1}$ to change $(2n)!$ to $(2n+1)!$. These give us, respectively, the following two formulas for the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}.$$

The first formula is particularly interesting when we connect it to the Problem 4, which counts paths in an $n \times n$ grid. Earlier in the semester, we found that the total number of paths from $(0,0)$ to (n,n) in this grid is $\binom{2n}{n}$; meanwhile, C_n is the number of paths that always stay above the main diagonal. The formula $C_n = \frac{1}{n+1}\binom{2n}{n}$ tells us that this is $\frac{1}{n+1}$ of the total number of paths.

3 Practice problems

These are optional, but if you'd like to think about the Catalan numbers (and the techniques we used in class) some more, here are some things you could think about.

1. We used Problem 3 to justify the recurrence for Catalan numbers. Explain why the number of solutions to Problems 1, 2, and 4 should also satisfy this recurrence.

2. Another way to prove that Problem 1, Problem 2, Problem 3, and Problem 4 all give us the Catalan numbers is by finding a bijection. Try finding some bijections between these problems! A good one to start with is to find a bijection between Problem 2 and Problem 4: a bijection that turns sequences of n opening and n closing parentheses into paths in the $n \times n$ grid that never go below the diagonal.

3. Prove that the following formula is also correct for the Catalan numbers: $C_n = \binom{2n}{n} - \binom{2n}{n+1}$.

4. In this lecture, we used the binomial theorem to expand $\sqrt{1-4x}$, interpreting it as $(1-4x)^{1/2}$. Something similar can be done to make sense of our generating function for counting multisets! Give a second proof of the expansion

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \left(\!\!\binom{n}{k}\!\!\right) x^k = \left(\!\!\binom{n}{0}\!\!\right) x^0 + \left(\!\!\binom{n}{1}\!\!\right) x^1 + \left(\!\!\binom{n}{2}\!\!\right) x^2 + \left(\!\!\binom{n}{3}\!\!\right) x^3 + \dots$$

   by using the binomial theorem.

5. If we define $(-\frac{1}{2})!$ to be $\sqrt{\pi}$, give a formula for $(n + \frac{1}{2})!$ in terms of n.

6. Suppose I have n pairs of ( ) parentheses and also n pairs of [ ] brackets. I would like to assemble them together into a sequence of length $4n$ that makes sense: each) closes a (and each] closes a [. For example, (([])[[()]]) would be a valid solution for $n = 3$.

   Find the number of ways to do this.

7. Consider a random walk on the number line: we start at 0, and at every time step, we go either left or right (adding -1 or $+1$) with equal probability.

   (a) What is the probability that at time $2n$, we return to 0 *for the first time*? (I am writing $2n$ and not n because it's impossible to return to 0 on an odd time step.)

   (b) Now suppose that our random walk is biased: it goes right (adding $+1$) with probability $\frac{2}{3}$ and left (adding -1) with probability $\frac{1}{3}$.

   How does the answer to (a) change?

   (c) If the answer to (a) or to (b) is summed over all $n \geq 1$, it gives us the probability that the random walk *ever* returns to 0.

   Find that probability—for both cases—by interpreting the infinite sum you get in terms of evaluating the generating function $C(x)$ at a specific value of x.