Enumerative Combinatorics Homework 2

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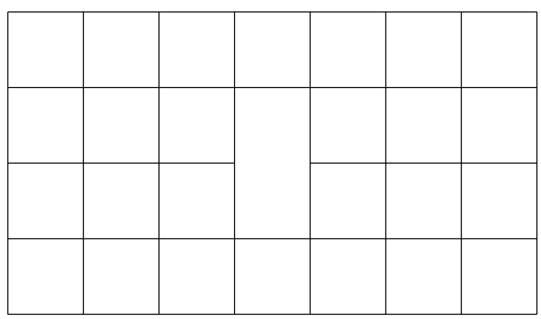
due Friday, January 26, 2024

1 Short answer

Here, just a number (or an expression using operations like $\binom{n}{k}$) is sufficient. (However, if you show your work, I can point out any mistakes you make.)

1. A figure skating contest has 12 participants. A panel of 12 judges evaluates the participants; each judge ranks the participants from best to worst, with no ties. In how many ways can all the rankings of all the judges be assigned?
2. How many rectangles are there in the picture below? (The missing line is intentional.)



3. How many subsets of the set $[13]$ include exactly one two-digit number: 10, 11, 12, or 13? (There is no restriction on how many one-digit numbers they include.)
4. In a class with 11 students, the teacher brings 5 identical apples to hand out as prizes. How many ways are there to distribute the apples, if no student should receive 3 or more apples?

2 Harder problems

For these problems, the answer is not just a single number, but a description or an explanation (which does not need to be long).

5. Describe a bijection between the following two sets:
   - The set of all pairs (A, B), where A and B are subsets of $[n]$, and additionally, A is a subset of B. (For example, if $n = 5$, the pair $(\{1, 3\}, \{1, 2, 3, 5\})$ is included.)
   - The set of all n-element sequences with values in $\{0, 1, 2\}$.

6. Let S be the set of all permutations of [8]. (To avoid confusion: for us, in this class, a permutation of [8] is a sequence of length 8 in which every element of [8] appears exactly once).

   We define an equivalence relations $\sim$ on S as follows. Let $(x_1, x_2, \ldots, x_8) \sim (y_1, y_2, \ldots, y_8)$ if, for each $i \in [8]$, x_i and y_i have the same parity: both are even and both are odd. For example, we have $(1, 2, 3, 4, 5, 6, 7, 8) \sim (3, 2, 1, 4, 5, 8, 7, 6)$, because both sequences alternate odd-even-odd-even-odd-even-odd-even.

   How many elements are in each equivalence class of $\sim$? Why is this number the same for every equivalence class? And how many equivalence classes are there?

7. **(This problem has been moved to the next homework assignment; feel free to think about it early if you like, but don't submit it as part of HW2.)**

   In a 25-student class, every lecture begins with a demonstration. For the demonstration, 5 students are chosen to come to the front of the room and do some kind of activity.

   How many lectures are required for us to be able to *guarantee* that no matter how the students are chosen, there is a pair of students that are part of the demonstration together more than once?

   Explain why your answer is correct. You do not have to prove that your answer is the best possible, but do not make the number unnecessarily large.