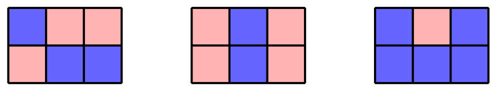
Enumerative Combinatorics Homework 3

Mikhail Lavrov

due Friday, February 9, 2024

1. How many ways are there to divide 9 people into three teams of equal size? (The only feature of a solution that matters is which people are on a team together; the teams have no identity of their own.)

2. Up to rotation, how many ways are there to color the cells of a 2×3 grid red and blue? (Some examples are shown below.)



3. In a 25-student class, every lecture begins with a demonstration. For the demonstration, 5 students are chosen to come to the front of the room and do some kind of activity.

   How many lectures are required for us to be able to *guarantee* that no matter how the students are chosen, there is a pair of students that are part of the demonstration together more than once?

   Explain why your answer is correct. You do not have to prove that your answer is the best possible, but do not make the number unnecessarily large.

4. Describe each of the following problems in terms of counting functions $[k] \to [n]$. State what n and k must be, and additional features of the counting problem: whether the function is required to be injective (one-to-one), or surjective (onto); whether we count equivalence classes up to permutations of $[k]$, or of $[n]$, or both.

   You do not have to solve the counting problems. (We have not necessarily covered all of them in class yet.)

   (a) Alice, Bob, and Carol are sharing a plate of 20 dumplings at a restaurant. We want to count the number of ways to decide how many dumplings each of them gets, with the restriction that we cannot leave any of them without dumplings entirely.

   (b) The number of solutions to $x + y + z = 10$, where x, y, z are integers and $0 \le x \le y \le z$.

   (c) The number of 8-character passwords where each character is an uppercase letter, and all 8 letters must be distinct. (This means "PASSWORD" would not be a valid password, but "CASEWORK" would be allowed.)

5. Give a combinatorial argument for the following identity:
$$\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3.$$
   (We can think of n^3 as counting sequences of 3 elements of $[n]$, or as functions from $[3]$ to $[n]$, but you should feel free to add flavor text to your argument and count the number of ways for 3 people to each pick a hat of one of n colors, or the number of ways to deliver a package, a letter, and a postcard to a street with n houses, or anything else you like.)