Enumerative Combinatorics Homework 4

Mikhail Lavrov

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Regarding Stirling numbers and Bell numbers, which may start appearing in your homework, I have two comments:

- Although $S(n,k)$ and $B(n)$ are unfamiliar functions, they are every bit as fundamental to us as $\binom{n}{k}$ or $n!$, so an answer like $S(10,3)$ counts as simplified to us, unless I say otherwise.
- While you're still becoming comfortable with computing these numbers, the "Stirling's triangle" in section 2.3 of the textbook (formula (2.7) on p. 71, at least in the edition I've got) is a useful resource for looking up $S(n,k)$ (and $B(n)$, with some work) for small n and k.

1. How many integer solutions are there to the equation $a+b+c+d+e=100$ that satisfy $a \ge 1$, $b \ge 2$, $c \ge 3$, $d \ge 4$, and $e \ge 5$?

2. (a) Find the number of functions $[k] \to [n]$ whose image has size j, up to permutations of $[k]$.

   In other words, find the number of ways to place k identical marbles into n labeled boxes, such that exactly j of the boxes have marbles in them.

   (b) Deduce a combinatorial identity by taking the sum of your answer to (a) over all j going from 0 to n. (Think about what that sum should count.)

3. How many 15-digit numbers are there which include every digit 0 through 9 at least once? (A sequence starting with 0 does not count as a 15-digit number.)

4. Find the number of expressions which are sums of products of the variables p, q, r, s, t, where each variable appears exactly once. Here are some examples:

$$p+q+r+s+t, \qquad pst+qr, \qquad pqrst.$$

5. On the previous assignment, you proved the combinatorial identity

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1}.$$

   The numbers $6, 6, 1$ have a connection to Stirling numbers. Find that connection, and use it to fill in the blanks in the identity

$$n^5 = \underline{\quad\quad}\binom{n}{5} + \underline{\quad\quad}\binom{n}{4} + \underline{\quad\quad}\binom{n}{3} + \underline{\quad\quad}\binom{n}{2} + \underline{\quad\quad}\binom{n}{1}.$$

   (For this problem, fill in the blanks with exact values, not formulas.)