Enumerative Combinatorics Homework 5

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1. Three pairs of siblings with the uncreative names $A_1, A_2, B_1, B_2, C_1, C_2$ want to sit in a row for a group photo. However, nobody wants to sit next to their sibling. Obviously. How many valid orders does that leave us with?

   (For example, $(A_1, B_1, C_1, A_2, B_2, C_2)$ is a valid order, but $(A_1, B_1, B_2, C_1, A_2, C_2)$ is not.)

2. In 11-dimensional $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$ tic-tac-toe, you win if you claim 4 cells that lie on a line. How many winning lines are there?

   ... There's a neat geometric solution to this problem, and if you have good intuition for 11-dimensional geometry, you can solve it from there. If you want a more combinatorial approach, you can think of winning lines in 11-dimensional tic-tac-toe as follows. These lines are given by taking an equation like

   $$\ell(t) = (1, 4, 1, t, t, 2, 3, 5 - t, t, 1, 5 - t)$$

   and plugging in $t = 1, 2, 3, 4$. The equation must have 11 components. Each component can be one of 1, 2, 3, 4, t, or $5 - t$. Either t or $5 - t$ must appear at least once. Finally, if we swap all t's for $5 - t$'s and vice versa, we get the same line, just parameterized in reverse.

3. The *golden ratio* is a fancy name for the number $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$. Its claim to fame is the identity $\varphi^2 = \varphi + 1$, which is something we'll find some use for in the near future.

   Write a proof by induction for the identity $\binom{n}{k} \leq \varphi^{n+k}$.

4. A sequence $(a_n)_{n=0}^{\infty}$ is defined by $a_0 = \frac{2}{3}$ and $a_n = 2^n - a_{n-1}$.

   Find the first few terms of the sequence, guess a formula, and prove it by induction.

5. In this problem, let $\exp(x)$ denote e^x (where $e \approx 2.178\ldots$ is Euler's number) and let $\lceil x \rceil$ be the ceiling of x: the smallest integer greater than or equal to x. (For example, $\lceil e \rceil = 3$.)

   Consider the formula $f(n) = \lceil \exp(\frac{n}{2} - 1) \rceil$. If we plug in $n = 1, 2, 3, 4, 5, 6, 7$ then we get $f(n) = 1, 1, 2, 3, 5, 8, 13$ which should begin to look somewhat familiar after the appropriate lecture.

   Is it true that $f(n)$ gives the n^{th} Fibonacci number for all n? I don't need a formal proof as the answer, but you should explain why it's true or false.